Linear Programming

INTRODUCTION

Linear programming was developed in 1947 by George B. Dantzig, Marshal Wood and their associates. It deals with the optimization (maximization or minimization) of a function of variables, known as objective functions. It is a set of linear equalities/inequalities known as constraint. Basically, linear programming is a mathematical technique, which involves the allocations of limited resources in an optimal manner on the basis of a given criterion of optimality. Linear programming is an optimization method applicable for the solution of problems in which the objective function and the constraints appear as linear functions of decision variables.

BASIC DEFINITIONS

1. Decision Variables

These are the variables, whose quantitative values are to be found from the solution of the model so as to maximize or minimize the objective function. The decision variables are usually denoted by $x_1, x_2, x_3, \ldots x_n$. It may be controllable or uncontrollable.

Controllable variables are those, whose values are under control of the decision makers. Uncontrollable variables are those, whose values are not under control.

2. Objective Function

It is the determinants of quantity either to be maximized or to be minimized. An objective function must include all the possibilities with profit or cost coefficient per unit of output. It is denoted by Z. The objective function can be stated as

$$\text{Max } Z \text{ or min } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

3. Constraints (Inequalities)

These are the restrictions imposed on decision variables. It may be in terms of availability of raw materials, machine hours, man-hours, etc.

$$a_{i1}\, x_1 + a_{i2}\, x_2 + a_{i3}\, x_3 + \ldots + a_{in}\, x_n\ (\le, =, \ge)\ b_i$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}\, x_1 + a_{m2}\, x_2 + a_{m3}\, x_3 + \ldots + a_{mn}\, x_n\ (\le, =, \ge)\ b_m$$

and $$x_1, x_2, x_3 \ldots x_n \ge 0 \tag{3}$$

Equation (1) is known as objective function.
Equation (2) represents the role of constants.
Equation (3) is non-negative restrictions.
Also $a'_{ij}s$ $b'_j s$ and $c'_j s$ are constants and $x'_j s$ are decision variables.
The above L.P.P. can be expressed in the form of matrix as follows:
Opt. $Z = CX$,
Subject to

$$AX\ (\le, =, \ge)\ B$$

and $X \ge 0$

where $$C = c_1, c_2, c_3 \ldots c_n$$
$$X = x_1, x_2, x_3 \ldots x_n$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n}$$

Example 1 A manufacturer produces two types of models M_1 & M_2. Each model of type M_1 requires 4 hr of grinding and 2 hr of polishing. Whereas model M_2 requires 2 hr of grinding and 5 hr of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works 60 hr a week and each polisher works 50 hr a week. Profit on model M_1 is Rs 4.00 and on model M_2 is Rs 5.00. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a weak? Formulate it as linear programming problem.

Solution

Decision Variables Let x_1 and x_2 be the number of units produced model M_1 and model M_2. Therefore, x_1 and x_2 be treated as decision variables.

Objective Function Since the profit on both the models is given and we have to maximize the profit. Therefore,

$$\text{Max } Z = 4x_1 + 5x_2 \qquad \ldots(1)$$

Constraints There are two constraints one for grinding and other for polishing. Two grinders are working. Therefore, number of hours available for grinding = $60 \times 2 = 120$ hours

Model M_1 requires 4 hr of grinding and Model M_2 requires 2 hours of grinding. Hence, the grinding constraint is given by

$$4x_1 + 2x_2 \le 120 \qquad \ldots(2)$$

There are 3 polishers. Total no. of hr available for polishing = $50 \times 3 = 150$ hr.

Model M_1 requires 2 hr of polishing, whereas model M_2 requires 5 hr of polishing. Therefore, we have

$$2x_1 + 5x_2 \le 150 \qquad \ldots(3)$$

Non-negative Restriction

$$x_1, x_2, \ge 0 \qquad \ldots(4)$$

From equations (1), (2), (3), and (4), we have

$$\begin{aligned} \text{Max } Z &= 4x_1 + 5x_2 \\ \text{S.T. } 4x_1 + 2x_2 &\le 120 \\ 2x_1 + 5x_2 &\le 150 \\ x_1, x_2, &\ge 0 \end{aligned}$$

Example 2 A paper mill produces two grades of papers X and Y. Because of raw material restrictions it cannot produce more than 500 tonnes of grade X and 400 tonnes of grade Y in a week. There are 175 production hr in a week. It requires 0.2 and 0.4 hr to produce one tonne of product X and Y respectively with corresponding profit of Rs 4.00 and 5.00 per tonne. Formulate the above as L.P.P. to maximize the profit.

Solution

Decision Variables Let x_1 and x_2 be the number of units of two grades of papers X and Y. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Function Since the profit of two grades of papers X and Y are given and we have to maximize the profit.

$$\therefore \quad \text{Max } Z = 400\, x_1 + 500\, x_2 \qquad \ldots(1)$$

Constraints There are two constraints one with respect to raw materials and other with respect to production hours.

$$\left.\begin{aligned} x_1 &\le 500 \\ x_2 &\le 400 \\ 0.2\, x_1 + 0.4\, x_2 &\le 175 \end{aligned}\right\} \qquad \ldots(2)$$

Example 4 A manufacturer produces three models I, II and III of a certain product. He uses two types of raw materials (A and B) of which 5000 and 8000 units respectively are available. Raw material of type A requires 3, 4 and 6 units of each model. Whereas type B requires 6, 4 and 8 of model I, II and III respectively. The labour time of each unit of model I is twice that of model II and three times of model III. The entire labour force of the factory can produce equivalent of 3000 units of model I. A market survey indicates that the minimum demand of three models is 600, 400 and 350 units respectively. However, the ratios of number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III are Rs 80, 50, and 120 respectively. Formulate this problem as linear programming model to determine the number of units of each product which will maximize the profit.

Solution

The above problem can be tabulated as given below:

Raw materials	Requirement per unit model			Quantity of raw material available (units)
	I	II	III	
A	3	4	6	5000
B	6	4	8	8000
Profit/unit (Rs)	80	50	120	
Proportion of labour time	1	$\frac{1}{2}$	$\frac{1}{3}$	Production equivalent of model I = 3000 units

Decision Variables Let x_1, x_2, x_3 be the number of units of models I, II and III respectively. Therefore, it will be treated as decision variables.

Objective Function Since profit per units of models are given and we have to maximize the profit. Therefore,

$$\text{Max } Z = 80x_1 + 50x_2 + 120x_3 \qquad \ldots(1)$$

Constraints As per the statement of problem constraints are given as (as per tabulated value)

$$\left.\begin{aligned} 3x_1 + 4x_2 + 6x_3 &\leq 5000 \\ 6x_1 + 4x_2 + 8x_3 &\leq 8000 \\ x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 &\leq 3000 \\ x_1 &\leq 600 \\ x_2 &\leq 400 \\ x_1 &\leq 350 \end{aligned}\right\} \qquad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, x_3 \geq 0 \qquad \ldots(3)$$

From equations (1), (2) and (3) finally, we have

$$\text{Max } Z = 80x_1 + 50x_2 + 120x_3$$

S·T·
$$x_1 \le 600$$
$$x_2 \le 400$$
$$x_3 \le 350$$
$$3x_1 + 4x_2 + 6x_3 \le 5000$$
$$6x_1 + 4x_2 + 8x_3 \le 8000$$
$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \le 3000$$
$$x_1, x_2, x_3 \ge 0$$

Example 5 A research laboratory has two melts *A* and *B* of copper (Cu), Nickel (Ni) and Zinc (Zn) alloy to make up a new alloy. The composition of metals are as follows.

Melt	Composition (Parts)		
	Cu	Ni	Zn
A	3	2	1
B	2	2	1

To make up a new alloy at least 15 kg of copper, 10 kg of nickel, and 6 kg of zinc are needed. Melt *A* cost Rs 45 per kg and melt *B* cost Rs 50 per kg. Formulate the L.P.P. for the quantities of each melt to be used to minimized cost.

Solution

The above data can be tabulated as follows.

Composition	Melt		Requirement of elements (Rs)
	A	*B*	
Cu	3	2	15
Ni	2	2	10
Zn	1	1	6
Cost per kg of melt (Rs)	45	50	

Decision Variables Let x_1 and x_2 be the quantity of melt *A* and *B* respectively. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Function Since cost per kg melt of product *A* and *B* are given and we have to minimize the cost. Therefore,

$$\text{Min } Z = 45x_1 + 50x_2 \qquad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\frac{3}{6}x_1 + \frac{2}{5}x_2 \ge 15$$

$$\left.\begin{array}{ll} & 5x_1 + 4x_2 \geq 150 \\ \text{or} & \frac{2}{6}x_1 + \frac{2}{5}x_2 \geq 10 \\ & 5x_1 + 6x_2 \geq 150 \\ \text{or} & \frac{1}{6}x_1 + \frac{1}{5}x_2 \geq 6 \\ & 5x_1 + 6x_2 \geq 180 \\ \text{or} & \end{array}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, >, 0 \quad \ldots(3)$$

From equation (1), (2) and (3), we have

$$\text{Min } Z = 45x_1 + 50x_2$$

$$\begin{aligned} \text{S.T.} \quad 5x_1 + 4x_2 &\geq 150 \\ 5x_1 + 6x_2 &\geq 150 \\ 5x_1 + 6x_2 &\geq 180 \\ x_1 + x_2 &\geq 0 \end{aligned}$$

Example 6 The objective of a diet problem is to ascertain the quantities of a certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The consideration is limited to milk, beef and eggs and to vitamines A, B and C. The number of milligrams of each of these vitamines contained within a unit of each food is given below.

Vitamin	Gallon of milk	Pound of beef	Dozen of eggs	Minimum daily requirement
A	1	1	10	1 mg
B	100	10	10	50 mg
C	10	100	10	10 mg
Cost	Rs 1.00	Rs 1.10	Rs 0.50	—

What is the L.P.P. for this problem?

Solution

Decision Variables Let the daily diet consist of x_1 gallons of milk, x_2 pounds of beef and x_3 dozens of eggs. Therefore, x_1, x_2 and x_3 can be treated as decision variables.

Objective Function Since cost per day of milk, beef and eggs are given and we have to minimize the total cost, therefore, we have

$$\text{Min } Z = 1.00x_1 + 1.10x_2 + 0.50x_3 \quad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\left.\begin{array}{rcl} x_1 + x_2 + 10x_3 & \geq & 1 \\ 100x_1 + 10x_2 + 10x_3 & \geq & 50 \\ 10x_1 + 100x_2 + 10x_3 & \geq & 10 \end{array}\right\} \quad \ldots(2)$$

Non-negative Restrictions ...(3)

$$x_1, x_2, x_3 \geq 0$$

From equation (1), (2) and (3), we have

$$\text{Min } Z = x_1 + 1.10x_2 + 0.50x_3$$

$$\begin{array}{ll} \text{S.T.} & x_1 + x_2 + 10x_3 \geq 1 \\ & 100x_1 + 10x_2 + 10x_3 \geq 50 \\ & 10x_1 + 100x_2 + 10x_3 \geq 10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Example 7 A firm can produce three types of cloth *A*, *B* and *C*. Three kinds of wool is required for it, say red, green and blue wools. One unit length of type *A* cloth needs 2 yards of red wool, 5 yards of blue wools, one unit length of type *B* cloth needs 3 yards of red wool, 4 yards of green wool, and 2 yards of blue wool, and one unit length of type *C* cloth needs 6 yards of green and 5 yards of blue wools. The firm has only a stock of 10 yards of red wool, 12 yards of green wool, and 17 yards of blue wool. It is assumed that the income obtained from one unit length of type *A*, *B* and *C* are Rs 4.00, 5.00 and 6.00 respectively. Determine how the firm should use the available material, so as to maximize the income from the finished cloths.

Solution

The above problem can be tabulated as:

Kinds of wool	Types of cloth			Stock of wool (yards)
	A	*B*	*C*	
Red	2	3	0	10
Green	0	4	6	12
Blue	5	2	5	17
Income from one unit of clothes (Rs)	4.00	5.00	6.00	

Decision Variables Let the firm produce x_1, x_2, x_3 yards of three types of cloth *A*, *B* and *C* respectively. Therefore, x_1, x_2 and x_3 can be treated as decision variables.

Objective Function Since the profit per unit length of type *A*, *B* and *C* are given and we have to maximize the profit, therefore, we have

$$\text{Max } Z = 4x_1 + 5x_2 + 6x_3 \quad \ldots(1)$$

Constraints As per the statement of given problem, we have

$$\left.\begin{array}{rcl} 2x_1 + 3x_2 + 0x_3 & \le & 10 \\ 0x_1 + 4x_2 + 6x_3 & \le & 12 \\ 5x_1 + 2x_2 + 5x_3 & \le & 17 \end{array}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1, x_2, x_3 \ge 0 \quad \ldots(3)$$

From equations (1), (2) and (3), we have

$$\text{Max } Z = 4x_1 + 5x_2 + 6x_3$$

$$\begin{array}{ll} \text{S.T.} & 2x_1 + 3x_2 + 0x_3 \le 10 \\ & 0x_1 + 4x_2 + 6x_3 \le 12 \\ & 5x_1 + 2x_2 + 5x_3 \le 17 \end{array}$$

and $$x_1, x_2, x_3 \ge 0$$

Example 8 An oil refinery uses blending process to produce gasoline in a typical manufacturing process. Crude A and B are mixed to produce gasoline G_1 and G_2. The input and output of the process are as follows:

Process	Input (tonnes)		Output (tonnes)	
	Crude A	Crude B	G_1	G_2
1	1	2	6	8
2	6	8	5	7

Availability of crude A is only 200 tonnes and B is 500 tonnes. Market demand of G_1 is 150 tonnes and G_2 is 200 tonnes. Profit on process 1 and process 2 is Rs 300 and 500 per tonne. What is the optimal mixture of two blending processes so that refinery can maximize its profit?

Solution

Decision Variables Let x_1 and x_2 be the number of tonnes to be produced by process 1 and process 2. Therefore, x_1 and x_2 can be treated as decision variables.

Objective Functions Since the profit on process 1 and process 2 is given and we have to maximize the profit. Therefore, we have

$$\text{Max } Z = 300x_1 + 500x_2 \quad \ldots(1)$$

Constraints As per the statement of problem, we have

$$\left.\begin{array}{rcl} 5x_1 + 6x_2 & \le & 200 \\ 2x_1 + 8x_2 & \le & 500 \\ 6x_1 + 5x_2 & \le & 150 \\ 8x_1 + 7x_2 & \le & 200 \end{array}\right\} \quad \ldots(2)$$

Non-negative Restrictions

$$x_1 \text{ and } x_2 \ge 0 \quad \ldots(3)$$

From equation (1), (2) and (3), we have

$$\text{Max } Z = 300x_1 + 500x_2$$

$$\begin{aligned} \text{S.T.}\quad 5x_1 + 6x_2 &\le 200 \\ 2x_1 + 8x_2 &\le 500 \\ 6x_1 + 5x_2 &\le 150 \\ 8x_1 + 7x_2 &\le 200 \\ x_1 \text{ and } x_2 &\ge 0 \end{aligned}$$

EXERCISE

1. A company produces two types of leather belts A and B A is of superior quality and B is of inferior quality. The respective profits are Rs 10 and Rs 5.00 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt A special type of buckle is required and 500 are available per day. There are 700 buckles available for belt B per day. Belt A needs twice as much as time as that required for belt B and the company can produce 500 belts if all of them were of the type A. Formulate L.P.P. for above problem.

Ans.
$$\text{Max } Z = 10x_1 + 5x_2$$

$$\begin{aligned} \text{S.T.}\quad x_1 + x_2 &\le 850 \\ x_1 &\le 500 \\ x_2 &\le 700 \\ 2x_1 + x_2 &\le 1000 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2. A company produces two types of caps. Each cap of the first type requires as much labour time as the second type. If all caps are of second type only; the company can produce a total of 500 caps a day. The market limits daily sales of the first and second type to 150 and 250 caps. Assume that the profit per cap are Rs 10 for type B. Formulate the problem as a linear programing model in order to determine the number of caps to be produced of each type as to maximize the profit.

Ans.
$$\text{Max } Z = 10x_1 + 5x_2$$

$$\begin{aligned} \text{S.T.}\quad 2x_1 + x_2 &\le 500 \\ x_1 &\le 150 \\ x_2 &\le 250 \\ x_1, x_2 &\ge 0 \end{aligned}$$

3. An oil refinery can blend three grades of crude oil to produce quality P and quality Q petrol. Two blending processes are available. For each production run the older process uses 5 units of crude A, 7 units of crude B and 2 units of crude C to produce 9 units of P and 7 units of Q. The newer processes uses 3 units of crude A. 9 units of crude B, and 4 units of crude C to produce 5 units of P and 9 units of Q petrol. Because of prior contract commitments the refinery must produce at least 500 units of P and 300 units of Q for the next month. It has available 1500 units of crude A, 1900 units of crude B and 1000 units of crude C. For each unit of P, the refinery receives Rs 60.00, while for each unit of Q it receives Rs 90.00. Find out the linear programming formulation of the problem to maximize the revenue.

Ans. $$\begin{aligned} \text{Max } Z &= 1170x_1 + 1110x_2 \\ \text{S.T.} \quad 5x_1 + 3x_2 &\le 150 \\ 7x_1 + 9x_2 &\le 1900 \\ 2x_1 + 4x_2 &\le 1000 \\ 9x_1 + 5x_2 &\le 500 \\ 7x_1 + 9x_2 &\le 300 \\ x_1, x_2 &\ge 0 \end{aligned}$$

4. Orient Paper Mill produces two grades of papers X and Y. Because of raw material restrictions not more than 400 tonnes of grade X and 300 tonnes of grade Y can be produced in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce one tonne of products X and Y respectively with corresponding profit of Rs 35.00 and Rs 60.00 per tonne. Formulate a linear programming model to optimize the product mixture for maximum profit.

Ans. $$\begin{aligned} \text{Max } Z &= 35X + 60Y \\ \text{S.T.} \quad 0.2X + 0.4Y &\le 160 \\ X &\le 400 \\ Y &\le 300 \\ X, Y, &\ge 0 \end{aligned}$$

5. Garima Enterprises manufactures three types of dolls. The boy requires $\frac{1}{2}$ metre of red cloth, $1\frac{1}{2}$ metre of green and $1\frac{1}{2}$ metre of black cloth and 5 kg of fibre. The girl requires $\frac{1}{2}$ metre of red cloth, 2 metre of green cloth and 1 metre of black and 6 kg of fibre. The dog requires of $\frac{1}{2}$ metre of red, 1 metre of green. $\frac{1}{4}$ metre of black, and 2 kg of fibre. The profit on the three are 3.00, 5.00 and 2.00 respectively. The firm has 1000 metres of red, 1500 metres of green, 2000 metre of black and 6000 kg of fibre. Set up a linear programming for maximum profit to find the number of dolls of each type to be manufactured.

Ans. $$\begin{aligned} \text{Max } Z &= 3x_1 + 5x_2 + 2x_3 \\ \text{S.T.} \quad 0.5x_1 + 0.5x_2 + 0.5x_3 &\le 1000 \\ 1.5x_1 + 2x_2 + x_3 &\le 1500 \\ 0.5x_1 + x_2 + 0.25x_3 &\le 2000 \\ 5x_1 + 6x_2 + 2x_3 &\le 6000 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

6. A resourceful home decorator manufactures two types of lamps say, P and Q. Both lamps go through two technicians first a cutter, second a finisher. Lamp A requires 2 hr of the cutters time, and 1 hour of finisher time. Lamp B requires 1 hours of cutters, 2 hours of finisher time. The cutter has 104 hours and finisher has 76 hours of available time each month. Profit on one lamps is Rs 6.00 and on one B Lamp is Rs 11.00. Assuming that he can sell all that he produces, how many of each types of lamp should be manufacturer to obtain the best return.

Ans. $$\begin{aligned} &\text{Max } Z = 6x_1 + 11x_2 \\ \text{S.T.}\quad & 2x_1 + x_2 \le 104 \\ & x_1 + 2x_2 \le 76 \\ & x_1, x_2 \ge 0 \end{aligned}$$

7. A firm manufactures two types of products A and B and sells them at a profit of Rs 2.00 on type A and Rs 3.00 on type B. Each product is processed on two machines M_1 and M_2. Type A requires one minute of processing time on M_1 and 2 minutes on M_2, type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hr 40 minutes, while machine M_2 is available for 10 hours during any working day. Formulate the problem as an L·P·P. and find how many products of each type should the firm produce each day in order to get maximum profit.

Ans. $$\begin{aligned} &\text{Max } Z = 2x_1 + 3x_2 \\ \text{S.T.}\quad & x_1 + x_2 \le 400 \\ & 2x_1 + x_2 \le 600 \\ & x_1, x_2 \ge 0 \end{aligned}$$

8. A cold drink plant has two bottling machines A and B. It produces and sells 8-ounce and 16-ounce bottles. The following data is available

Machine	8-ounce	16-ounce
A	100/minute	40/minute
B	60/minute	75/minute

The machines can be run 8 hr per day 5 days per week. Weekly production of the drinks cannot exceed 3,00,000 ounces and the market can absorb 25,000 eight-ounce bottles and 7000 sixteen-ounce bottles per week. Profit on these bottles as 35 paise and 25 paise per bottle respectively. The planner wishes to maximize his profit subject to all the production and marketing restrictions. Formulate it as an L·P·P.

Ans. $$\begin{aligned} &\text{Max } Z = 0.35x_1 + 0.25x_2 \\ \text{S.T.}\quad & 8x_1 + 16x_2 \le 300000 \\ & 2x_1 + 5x_2 \le 480{,}000 \\ & 5x_1 + 4x_2 \le 720{,}000 \\ & x_1 \le 25000 \\ & x_2 \le 7000 \\ & x_1, x_2 \ge 0 \end{aligned}$$

9. A company manufactures two products A and B. These products are processed in the same machine. It takes 10 minutes to process one unit of product A and 2 minutes for each unit of product B and the machine operates for a maximum of 35 hr in a week. Product A requires 1.0 kg and B 0.5 kg of raw material per unit the supply of which is 600 kg per week. Product A costs Rs 5.00 per unit and sold at Rs 10. Product B costs Rs 6.00 per unit and can be sold in the market at a unit price of Rs 8.00. Determine the number of units per week to maximize the profit.

Ans. $$\text{Max } Z = 5x_1 + 2x_2$$
$$\text{S.T.} \quad 10x_1 + 2x_2 \leq 2100$$
$$x_1 + 0.5x_2 \leq 600$$
$$x_2 \leq 800$$
$$x_1, x_2 \geq 0$$

10. An electric appliance company produces two products: refrigerators and ranges. Production takes place in two separate departments I and II. Refrigerators are produced in department 1 and ranges in department II. There products are sold on weekly basis. The weekly production cannot exceed 25 refrigerators, and 35 ranges. The company regularly employs a total of 60 workers in two departments. A refrigerator requires 2 man weeks labour while a range requires 1 man week labour. A refrigerator contributes a profit of Rs 60.00 and a range contributes a profit of Rs 40.00. How many units of refrigerators and ranges should the company produce to realize the maximum profit. Formulate the above as an L.P.P.

Ans. $$\text{Max } Z = 60x_1 + 40x_2$$
$$\text{S.T.} \quad x_1 \leq 25$$
$$x_2 \leq 35$$
$$2x_1 + x_2 \leq 60$$
$$x_1, x_2 \geq 0$$

GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEMS

Graphical method is applicable to find the simple linear programming problem with two decision variables. Various steps for solving the problems are given below:

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph such that each will geometrically respect a straight line.
3. Identify the feasible region. If the inequality constraint corresponding to that line is ≤, then the region below the line in the first quadrant is to be shaded. For the inequality constraint ≥, then the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region.
4. Locate the corner points of the feasible region.
5. Draw the straight line to represent the objective function.
6. Test the objective function at each corner point of the feasible region and choose the point, where objective function obtains optimal value.

Example 9 Solve the following L.P.P. by graphical method

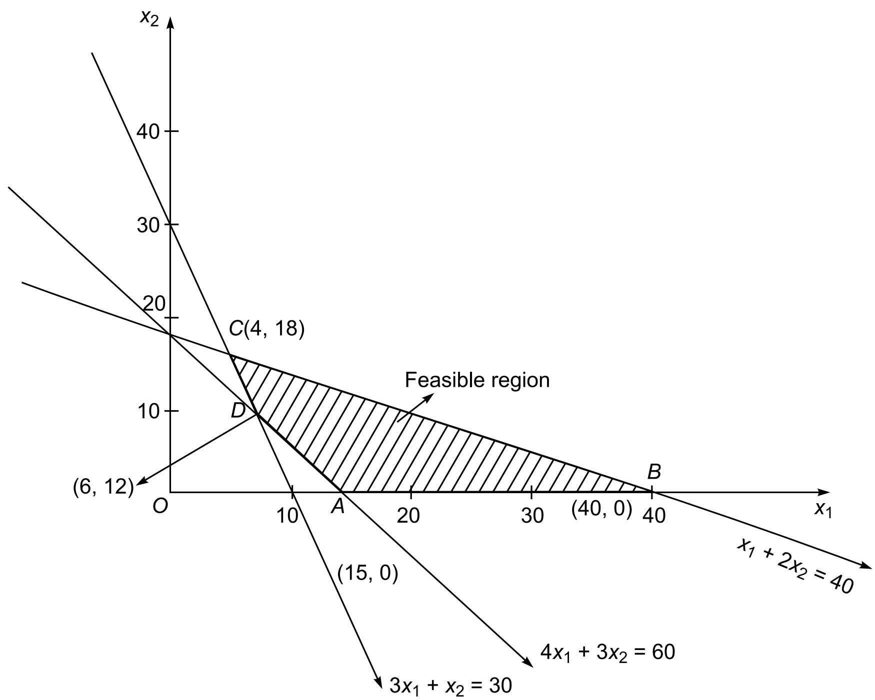
$$\text{Min } Z = 20x_1 + 10x_2$$
$$\text{Subject to } x_1 + 2x_2 \leq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Solution

Convert all the inequalities of the constraints into equations, we have

$$x_1 + 2x_2 = 40$$
$$3x_1 + x_2 = 30$$
$$4x_1 + 3x_2 = 60$$
$$x_1 + 2x_2 = 40 \text{ passes through } (0, 20)\ (40, 0)$$
$$3x_1 + x_2 = 30 \text{ passes through } (0, 30)\ (10, 0)$$
$$4x_1 + 3x_2 = 60 \text{ passes through } (0, 20)\ (15, 0)$$

Plot above equations on graph, we have



Here feasible region is *ABCD*.

The coordinates of *ABCD* are $A(15, 0)$ $B(40, 0)$, $C(4, 18)$, $D(6, 12)$

Now

Corner Points	Coordinate	Value of Z
A	(15, 0)	300
B	(40, 0)	800
C	(4, 18)	260
D	(6, 12)	240

Therefore, minimum value of Z occurs at $D(6, 12)$. Hence, optimal solution is $x_1 = 6, x_2 = 12$.

Example 10 Solve the following L.P.P. using graphical methods

$$\text{Max } Z = 6x_1 + 8x_2$$
$$\text{Subject to } 5x_1 + 10x_2 \leq 60$$
$$4x_1 + 4x_2 \leq 40$$
$$x_1, x_2 \geq 0$$

Solution

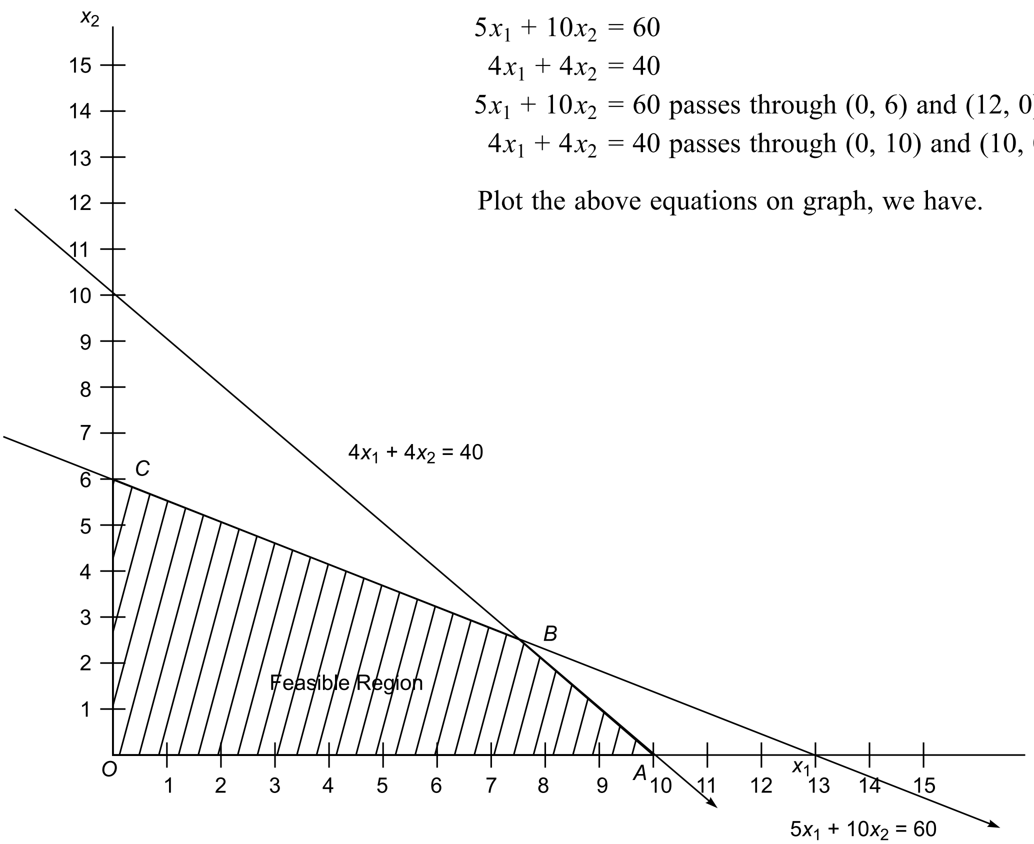
Convert all the equalities of the constraint into equations, we have

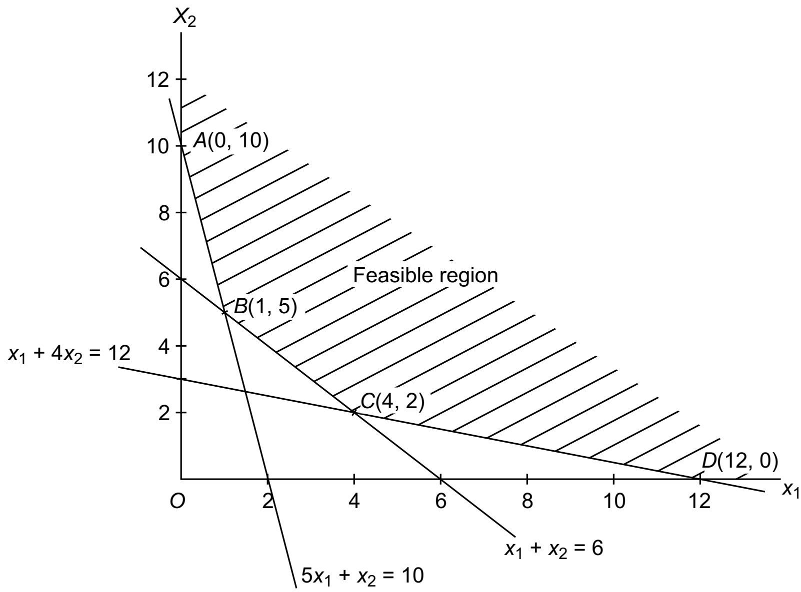
$$5x_1 + 10x_2 = 60$$
$$4x_1 + 4x_2 = 40$$

$5x_1 + 10x_2 = 60$ passes through (0, 6) and (12, 0)

$4x_1 + 4x_2 = 40$ passes through (0, 10) and (10, 0)

Plot the above equations on graph, we have.





Now the coordinates of points *ABCD* are $A(0, 10)$, $B(1, 5)$, $C(4, 2)$, $D(12, 0)$

Corner Points	Coordinate	Value of Z
A	(0, 10)	20
B	(1, 5)	13
C	(4, 2)	16
D	(12, 0)	36

Hence, minimum value occurs at point $B(1, 5)$. Therefore, optimum solution is given by

$$x_1 = 1, x_2 = 5 \text{ and } \min Z = 13.$$

Example 12 Solve the following L.P.P. by graphical method

$$\text{Max } Z = 3x_1 + 2x_2$$

$$\text{S.T.} \quad x_1 - x_2 \geq 1$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solution

Convert the inequality constraints into equations. We have

$$x_1 - x_2 = 1$$

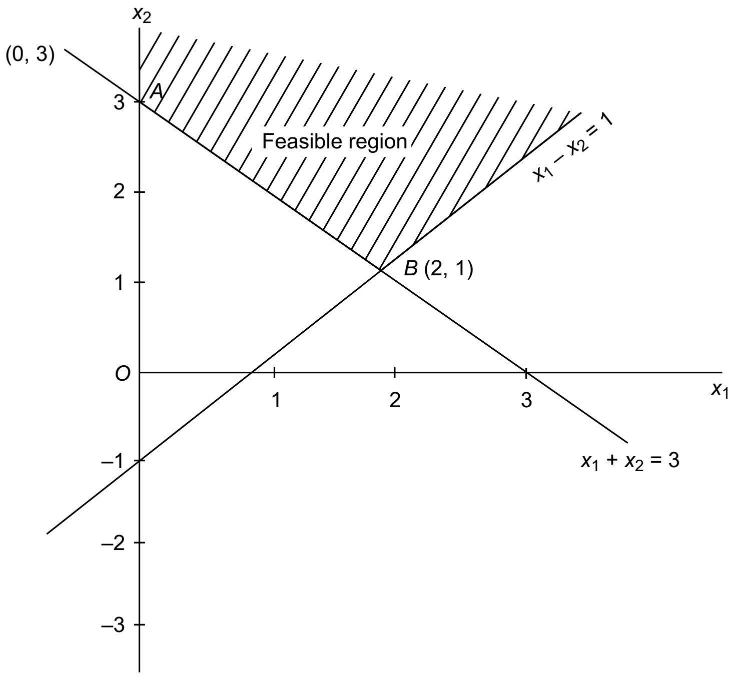
$$x_1 + x_2 = 3$$

Now

$$x_1 - x_2 = 1 \text{ passes through } (0, -1) \text{ and } (1, 0)$$

$$x_1 + x_2 = 3 \text{ passes through } (0, 3) \text{ and } (3, 0)$$

Plot above equations on graph, we have



Here the solution space is unbounded. The value of objective function at the vertices A and B are $Z(A) = 6$, $Z(B) = 6$. But there exists points in the convex region for which the value of the objective function is more than 8. In fact, the maximum value of Z occurs at infinity. Hence, the problem has an unbounded solution.

Example 13 By graphical method solve the following

$$\text{Max } Z = 3x_1 + 4x_2$$

$$\begin{aligned} \text{S.T. } 5x_1 + 4x_2 &\leq 200 \\ 3x_1 + 5x_2 &\leq 150 \\ 5x_1 + 4x_2 &\geq 100 \\ 8x_1 + 4x_2 &\geq 80 \\ x_1, x_2 &\geq, 0 \end{aligned}$$

Solution

Convert the inequality constraints into equations, we have

$$\begin{aligned} 5x_1 + 4x_2 &= 200 \\ 3x_1 + 4x_2 &= 100 \\ 8x_1 + 4x_2 &= 80 \\ 3x_1 + 5x_2 &= 150 \end{aligned}$$

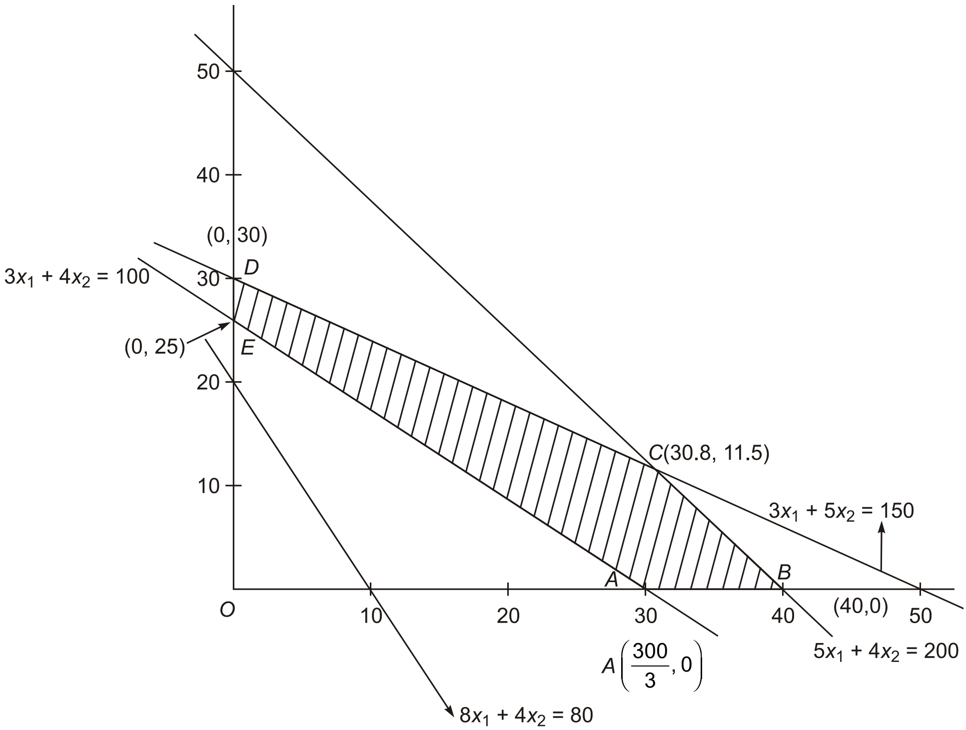
Now $\quad 5x_1 + 4x_2 = 200$ passes through (0, 50) and (40, 0)

$3x_1 + 4x_2 = 100$ passes through (0, 25) and $\left(\frac{100}{3}, 0\right)$

$8x_1 + 4x_2 = 80$ passes through (0, 20) and (10, 0)

$3x_1 + 5x_2 = 150$ passes through (0, 30) and (50, 0)

Plot the above equations on graph, we have



Here feasible region is *ABCDE*. Coordinates are given by $A\left(\frac{100}{3}, 0\right)$, $B(40, 0)$, $C(30.8, 11.5)$, $D(0, 30)$ and $E(0, 25)$.

Corner points	Coordinate	Value of Z
A	$\left(\frac{100}{3}, 0\right)$	100
B	(40, 0)	120
C	(30.8, 11.5)	138.4
D	(0, 30)	120
E	(0, 25)	100

6.

$$\begin{aligned} \text{Max } Z &= 3x_1 - 2x_2 \\ \text{S.T.} \quad x_1 + x_2 &\le 1 \\ 2x_1 + 2x_2 &\ge 6 \\ 3x_1 + 2x_2 &\ge 48 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7. [**Ans.** No feasible solution]

$$\begin{aligned} \text{Min } Z &= 3x_1 - 2x_2 \\ \text{S.T.} \quad x_1 + x_2 &\le 1 \\ 2x_1 + 2x_2 &\ge 6 \\ 3x_1 + 2x_2 &\ge 48 \\ x_1, x_2 &\ge 0 \end{aligned}$$

[**Ans.** No feasible solution]

8. A company produces two different products A and B. The company makes a profit of Rs 40 and Rs 30 per unit on A and B respectively. The production process has a capacity of 30,000 man hours. It takes 3 hr to produce one unit of a A and one hr to produce one unit of B. The market survey indicates that the maximum number of units of product A that can be sold is 8000 and those of B is 12,000 units. Formulate the problem and solve it by graphical method to get maximum profit.

Ans.

$$\begin{aligned} \text{Max } Z &= 40x_1 + 30x_2 \\ \text{S.T.} \quad 3x_1 + x_2 &\le 30{,}000 \\ x_1 &\le 8000 \\ x_2 &\le 12000 \\ x_1, x_2 &\ge 0 \end{aligned}$$

[**Ans.** $x_1 = 6000$, $x_2 = 1200$, Max $Z = 600000$]

SIMPLEX METHOD

It is an iterative procedure for solving an L.P.P. in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the successing vertex is less or more as the case may be more than its process vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solutions. It is applicable for any number of decision variables.

Basic Terms Involved in Simplex Method

1. Standard Form of an L.P.P. In standard form of the objective function, namely, maximize or minimize, all the constraints are expressed as equations moreover R.H.S. of each constraint and all variables are non-negative.

2. Slack Variables These variables are added to less than or equal to type constraints to change it into equality.

3. Surplus Variables These variables are substrates from a greater than or equal to type constraint to change it into equality.

4. Basic Solution Given a system of m linear equations with n variables ($m < n$). Any solution which is obtained by solving for m variables keeping the remaining (n–m) variables zero is called a basic solution.

5. Basic Feasible Solution A basic solution, which also satisfies the non-negative constraints, is called basic feasible solution.

6. Non-Degenerate Basic Solution It is the basic feasible solution, which has exactly m positive, i.e., none of basic variables are zero.

7. Degenerate Basic Feasible Solution A B.F.S. is said to be degenerate if one or more basic variables are zero.

8. Feasible Solution Any solution to an L.P.P. which satisfies the non-negative restrictions, is called feasible solution.

9. Optimal Solution A basic feasible solution of an L.P.P. which gives optimum value of the objective function is called optimal solution.

10. Unbounded Solution If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

11. Canonical Form In canonical form, if the objective function is of maximization, all the constraints other than non-negative conditions are $\leq$ type. If the objective function is of minimization, all the constraints other than non-negative conditions are $\geq$ type.

Simplex Algorithm

The various steps for the computation of an optimum solution by simplex method are as follows:

1. Check whether the objective function of a given L.P.P. is to be maximized or minimized. If it is to be minimized, then convert into maximization case.
2. Check whether all b_i (i = 1, 2, 3 … n) are positive or not. If any one b_i is negative then make it positive by multiplying –1 in equation of the constraint.
3. Express the problem in standard form by introducing stack/surplus variables to convert the inequality into equation.
4. Find an initial basic feasible solution to the problem and put it in the simplex table.
5. Prepare the initial simplex table.

	Initial simplex table						
		C_j	x_1	x_2	…	x_n	0 0 0
C_B	B	X_B	x_1	x_2	…	x_n	$S_1,\ S_2 \dots S_m$
C_{B1}	S_1	b_1	a_{11}	a_{12}	…	a_{1n}	1 0 0
C_{B2}	S_2	b_2	a_{21}	a_{22}	…	a_{2n}	0 1 0

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2	$\text{Min}\left(\frac{X_B}{x_1}\right)$
0	S_1	4	1	1	1	0	$\frac{4}{1} = 4$
$\leftarrow$ 0	S_2	2	$\boxed{1}$	–1	0	1	$\frac{2}{1} = 2$
	Z_j	0	0	0	0	0	
		$Z_j - C_j$	–3 $\uparrow$	–2	0	0	

Since all the values of $Z_j - C_j$ is not positive. Therefore, initial basic feasible solution is not optimum. To find optimum solution select the most negative value of $Z_j - C_j$. Here –3 is the most negative value of $Z_j - C_j$. It will enter in the basis and treated as entering variable and corresponding column will known as key column.

Now find leaving variable by taking min $\left(\frac{X_B}{x_1}, x_1 > 0\right)$. Here, minimum value exists in the second row, therefore, it will be treated as key row and S_2 will leave the basis.

Find key element by intersection of key row and key column. Here key element is $\boxed{1}$. Now make all other elements of key column to zero by taking matrix row transformation $R_1 \rightarrow R_1 - R_2$ and prepare the new simplex table.

First simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2
$\leftarrow$ 0	S_1	2	0	$\boxed{2}$	1	–1
3	x_2	2	1	–1	0	0
	Z_j	6	3	–3	0	0
		$Z_j - C_j$	0	–5 $\uparrow$	0	0

Further, all values of $Z_j - C_j$ is not positive. Therefore solution is not optimal. Here –5 is the most negative number and it will enter in the basic. Corresponding column is treated as key column. Key row is the first row. S_1 will leave the basis. Key element is not unity. Make it unity and then apply $R_2 \rightarrow R_2 + R_1$ to make all after element of key column to zero. Now form the second simplex table.

Second simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2
2	x_2	1	0	1	1/2	–1/2

Contd.

3	x_1	3	1	0	1/2	1/2
	Z_j	11	3	2	5/2	1/2
		$Z_j - C_j$	0	0	5/2	1/2

Here all the values of $Z_j - C_j$ are positive. Hence, optimum solution will exist and it is given by Max $Z = 11$, $x_1 = 3$, $x_2 = 1$

Example 15 Solve the L.P.P. by simplex method

$$\begin{aligned} \text{Max } Z &= 10x_1 + 6x_2 \\ \text{S.T.} \quad x_1 + x_2 &\leq 2 \\ 2x_1 + x_2 &\leq 4 \\ 3x_1 + 8x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution

Given problem is the case of maximization. Also values of b_1 b_2 and b_3 are positive. By introducing the slack variables S_1, S_2 and S_3 convert the problem into standard form.

$$\begin{aligned} \text{Max } Z &= 10x_1 + 6x_2 + 0S_1 + 0S_2 + 0S_3 \\ \text{S.T.} \quad x_1 + x_2 + S_1 &= 2 \\ 2x_1 + x_2 + S_2 &= 4 \\ 3x_1 + 8x_2 + S_3 &= 12 \\ x_1, x_2, S_1, S_2, S_3 &\geq 0 \end{aligned}$$

Initial basic feasible solution is given by $x_1 = 0$, $x_2 = 0$, $S_1 = 2$, $S_2 = 4$, $S_3 = 12$

Initial simplex table is given by

Initial simplex table

		C_j	10	6	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
← 0	S_1	2	$\boxed{1}$	1	1	0	0
0	S_2	4	2	1	0	1	0
0	S_3	12	3	8	0	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–10 ↑	–6	0	0	0

Here all the values of $Z_j - C_j$ is not positive.

∴ Optimal solution will not exist. To find optimal solution select the most negative values of $Z_j - C_j$. Here –10 is the most negative number. It will enter in the basis. Corresponding column is treated as key column. Find the key row by taking

Now prepare initial simplex table.

Initial simplex table

		C_j	1	1	3	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
0	S_1	2	3	2	1	1	0
$\leftarrow$ 0	S_2	2	2	1	$\boxed{2}$	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–1	–1	–3 $\uparrow$	0	0

Here all the values of $Z_j - C_j$ are not positive. Hence, the solution is not optimum. To find the optimum solution, select the most negative value of $Z_j - C_j$. Here –3 is the most negative value. It will enter in the basis. Corresponding column is treated as key column. To find the key row, find

$$\min\left(\frac{X_B}{x_3}, x_3 > 0\right) = \min\left(\frac{2}{1}, \frac{2}{2}\right)$$

$$= \min(2, 1) = 1$$

Hence, S_2 will leave the basis.

$\boxed{2}$ is the key element, because it is intersection of key row and key column. Make it unity and then apply matrix row transformation to make other element of key column to zero. Taking $R_1 \to R_2 - R - 1$, we have.

First simplex table.

		C_j	1	1	3	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
0	S_1	1	2	3/2	0	1	–1
3	x_3	1	1	1/2	1	0	1/2
	Z_j	3	1	3/2	3	0	3/2
		$Z_j - C_j$	0	1/2	0	0	3/2

Here, all the values of $Z_j - C_j$ are positive. Hence, optimum solution will exist and it is given by $x_1 = x_2 = 0$, $x_3 = 1$ and

Max $Z = 3$. **Ans**

Example 17 Use simplex method to solve the L.P.P.

$$\text{Min } Z = x_1 - 3x_2 + 2x_3$$

$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 \le 7.$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

Solution

Given problem is the ease of minimization. We shall convert it into maximization.

$$\text{Max } Z = -x_1 + 3x_2 - 2x_3$$
$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 \le 7.$$
$$-2x_1 + 4x_2 + 0x_3 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

By introducing the slack variables S_1, S_2 and S_3, we convert inequality into equality, i.e.,

$$\text{Max } Z = -x_1 + 3x_2 - 2x_3 + 0S_1 + 0S_2 + 0S_3$$
$$\text{S.T.} \quad 3x_1 - x_2 + 2x_3 + S_1 = 7$$
$$-2x_1 + 4x_2 + 0x_3 + S_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$
$$x_1, x_2, x_3\ S_1, S_2, S_3 \ge 0$$

Initial basic feasible solution is given by $x_1 = x_2 = x_3 = S_1 = S_2 = S_3 = 0$

Now prepare the initial simplex table, we have

Initial simplex table

		C_j	–1	3	–2	0	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
0	S_1	7	3	–1	2	1	0	0
← 0	S_2	12	–2	$\boxed{4}$	0	0	1	0
0	S_3	10	–4	3	8	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	–1	–3	2	0	0	0
					↑			

Here all values of $Z_j - C_j$ are not positive. To find optimum solution select most negative values of $Z_j - C_j$ i.e., –3 is most negative number. Corresponding column will be key column. To find key row, we have

$$\min\left(\frac{X_B}{x_2}\right) = \min\left(\frac{7}{-1}, \frac{12}{4}, \frac{10}{3}\right) = 3$$

$\boxed{4}$ is the key element and corresponding row is key row. Make the key element. Unity and all other elements of key column zero by matrix row transformation. Taking $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 3R_2$, we have the first simplex table.

$$x_1 + 4x_2 + 0x_3 \leq 420$$
$$x_1, x_2, x_3 \geq 0$$

Solution

The given problem is ease of maximization all values of b_i's (i = 1, 2, 3) are positive. By introducing slack variables convert the problem into standard form and inequality into equality; we have

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$$
$$\text{S.T.} \quad x_1 + 2x_2 + x_3 + S_1 = 430$$
$$3x_1 + 0x_2 + 2x_3 + S_2 = 460$$
$$x_1 + 4x_2 + 0x_3 + S_3 = 420$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Initial basic feasible solution is given by $x_1 = x_2 = x_3 = 0$

$$S_1 = 430, S_2 = 460, S_3 = 420$$

Now prepare initial simplex table.

Initial simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3
0	S_1	430	1	2	1	1	0	0
← 0	S_2	460	3	0	$\boxed{2}$	0	1	0
0	S_3	420	1	4	0	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	–3	–2	–5 ↑	0	0	0

Here all values of $Z_j - C_j$ are not positive. Hence, solution is not optimum. To find optimum solution select the most negative number. Here –5 is the most negative number it will enter in basis. Corresponding column will be treated as key column. To find key row, find min $\left(\frac{X_B}{x_3}\right)$

$$= \min = \left(\frac{430}{1}, \frac{460}{2}, \frac{420}{0}\right)$$
$$= 230$$

∴The basic variable S_2 will leave the basis.

$\boxed{2}$ is the key element make it unity and other element of key column zero by matrix row transformation. Now we have first simplex table.

First simplex table

C_B	B	C_j X_B	3 x_1	2 x_2	5 x_3	0 S_1	0 S_2	0 S_3
0	S_1	200	$\frac{-1}{2}$	$\boxed{2}$	0	1	1/2	0
5	x_3	230	$\frac{3}{2}$	0	1	0	1/2	0
0	S_3	420	1	4	0	0	0	1
	Z_j	1150	15/2	0	5	0	5/2	0
		$Z_j - C_j$	$\frac{9}{2}$	–2 ↑	0	0	5/2	0

Further, all values of $Z_j - C_j$ are not positive. Hence, repeat the above process.

$\boxed{2}$ is the key element. Make it unity and other elements of key column zero by applying matrix row transformation, we have the second simplex table.

Second simplex table

C_B	B	C_j X_B	3 x_1	2 x_2	5 x_3	0 S_1	0 S_2	0 S_3
2	x_2	100	–1/4	1	0	1/2	–1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	s_3	20	2	0	0	–2	1	1
	Z_j	1350	7	2	5	1	2	0
		$Z_j - C_j$	4	0	0	1	2	0

Since all values of $Z_j - C_j \geq 0$. Hence, the solution is optimum. It is given by

$$x_1 = 0,\ x_2 = 100,\ x_3 = 230,\ \text{Max } Z = 1350 \quad \textbf{Ans}$$

EXERCISE

Solve the Following L.P.P. by simplex method.

1.
$$\text{Max } Z = x_1 + 2x_2 + x_3$$
$$\text{S.T.} \quad 2x_1 + x_2 - x_3 \geq -2$$
$$-2x_1 + x_2 - 5x_3 \leq 6$$
$$4x_1 + x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = 0$, $x_2 = 4$, $x_3 = 2$ Max $Z = 10$]

ARTIFICIAL VARIABLE TECHNIQUES

If some constraints are of '=' or ≥ type, then they will not contain any basic variables. In such eases, we introduce a new variable called artificial variable. These variables are fictitious and cannot have any physical meaning. Artificial variable is merely a device to get the starting basic feasible solution. To solve the L.P.P. there are two methods:

(i) The Big M method (Method of penalties)
(ii) The Two-phase simplex method.

Big M Method

To solve the L.P.P. by Big M. method, various steps are given below.

(i) Express the problem in standard form by introducing slack variables, surplus variables and artificial variables as required in the problem.
(ii) Add non-negative artificial variables to the left side of each of the equations corresponding to constraint of the type =, and ≥. These variables do not appear in the final solution. This is achieved by assigning very large penalty (–M for maximization) in the objective function.
(iii) Solve the modified L.P.P. by simplex method until any one of three cases may arise:

1. If no artificial variables appears in the basis and optimality conditions of simplex method is satisfied, then initial solution is an optimum basic feasible solution.
2. If at least one artificial variable appears in the optimum basis at zero level and the optimality conditions of simplex method are satisfied, then the current solution is an optimum basic feasible solution.
3. If at least one artificial variable appears in the basis at positive level and optimality condition of simplex method is satisfied, then the problem has no feasible solution.

Example 19 Use Big. M Method to solve the following L.P.P.

$$\text{Max } Z = 3x_1 + 2x_2$$
$$\text{S.T. } 2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution

Express the problem into standard form by introducing slack variables, surplus variable and artificial variable, we have

$$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$$

$$\text{S.T. } 2x_1 + x_2 + S_1 = 2$$
$$3x_1 + 4x_2 - S_2 + A_1 = 12$$
$$x_1, x_2, S_1, S_2, A_1 \geq 0$$

The initial basic feasible solution is given by

$$x_1 = x_2 = 0,\ S_1 = 2,\ A_1 = 12$$

Now solve the above L.P.P. by general simplex method. Form the initial simplex table.

Initial simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2	$-M$ / A_1
$\leftarrow$ 0	S_1	2	2	$\boxed{1}$	1	0	0
$-M$	A_1	12	3	4	0	-1	1
	Z_j	$-12M$	$-3M$	$-4M$	0	M	$-M$
		$Z_j - C_j$	$-3M - 3$	$-4M - 2$ $\uparrow$	0	M	0

Here, all the values of $Z_j - C_j$ are not positive hence, optimality condition of simplex method is not satisfied. To find the optimum solution, select the most negative number of $Z_j - C_{j.}$ Here $-4M - 2$ is the most negative number. Corresponding column is treated as key column. To find key row, find min $\left(\dfrac{X_B}{x_2}\right) = \min\left(\dfrac{2}{1}, \dfrac{12}{4}\right) = (2, 3) = 2.$

$\boxed{1}$ is key element. Therefore, S_1 will leave the basis. Make all elements of key column zero by applying matrix row transformation. i.e., $R_2 \to R_2 - 4R_1$.

Now we have the first simplex table.

First simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	0 / S_1	0 / S_2	$-M$ / A_1
2	x_2	2	2	1	1	0	0
$-M$	A_1	4	-5	0	-4	-1	1
	Z_j	$4 - 4M$	$4 + 5M$	2	$2 + 4M$	M	$-M$
		$Z_j - C_j$	$5M + 1$	0	$2 + 4M$	M	0

Here, all values of $Z_j - C_j \geq 0$. Hence, optimality condition of simplex method is satisfied. Also one artificial variable appears in the optimum basis at positive level, therefore the given L.P.P. will have no feasible solution.

Example 20 Solve the following L.P.P., using Big. M. method

$$\begin{aligned} \text{Max } Z &= x_1 + 2x_2 + 3x_3 - x_4 \\ \text{S.T.} \quad & x_1 + 2x_2 + 3x_3 = 15 \\ & 2x_1 + x_2 + 5x_3 = 20 \\ & x_1 + 2x_2 + x_3 + x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Solution

Here all constraints are in the form of equality. Therefore, we introduce artificial variables A_1, A_2 to convert the problem into standard form.

$$\begin{aligned}
\text{Max } Z &= x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3 \\
\text{S.T.} \quad x_1 + 2x_2 + 3x_3 + A_1 &= 15 \\
2x_1 + x_2 + 5x_3 + A_2 &= 20 \\
x_1 + 2x_2 + x_3 + x_4 &= 10 \\
x_1, x_2, x_3, x_4 &\geq 0 \\
A_1, A_2, A_3, &\geq 0
\end{aligned}$$

The initial solution is given by

$$x_1 = x_2 = x_3 = 0, A_1 = 15, A_2 = 20, x_4 = 10$$

Now prepare the initial simplex table, which is given by

Initial simplex table

C_B	B	C_j / X_B	1 / x_1	2 / x_2	3 / x_3	–1 / x_4	$-M$ / A_1	$-M$ / A_2
$\leftarrow -M$	A_1	15	1	2	3	0	1	0
$-M$	A_2	20	2	1	$\boxed{5}$	0	0	1
–1	x_4	10	1	2	1	1	0	0
	Z_j	$-35M - 10$	$-3M - 1$	$-3M - 2$	$-8M - 1$	–1	$-M$	$-M$
		$Z_j - C_j$	$-3M - 2$	$-3M - 4$	$-8M - 4$ $\uparrow$	0	0	0

Here optimality condition of simplex method is not satisfied, i.e., all the values of $Z_j - C_j$ are not positive. To find the optimum value, select the most negative value of $Z_j - C_j$, i.e., $-8M - 4$ will enter in the basis. It is treated as key column. To find key row, find min $\left(\frac{X_B}{x_3}\right)$

$$= \min\left(\frac{15}{3}, \frac{20}{5}, \frac{10}{1}\right) = \min(5, 4, 10) = 4$$

Hence, artificial variable A_2 will leave the basis.

$\boxed{5}$ is treated as key element. Make it unity and also other element of key column to zero by taking matrix row transformation.

Now we proceed for the first simplex table.

First simplex table

C_B	B / A_1	C_j / X_B	1 / x_1	2 / x_2	3 / x_3	–1 / x_4	$-M$ / A_1
$\leftarrow -M$	x_3	3	–1/2	$\boxed{7/5}$	0	0	1
3	x_4	4	2/5	1/5	1	0	0
–1	Z_j	6	3/5	9/5	0	1	0

Contd.

		$-3M+6$	$\frac{1}{3}M+\frac{3}{5}$	$\frac{-7}{5}M-\frac{6}{5}$	3	–1	$-M$
		$Z_j - C_j$	$\frac{1}{5}M\frac{-2}{5}$	$\frac{-7}{5}M-\frac{16}{5}$ ↑	0	0	0

Further optimality condition is not satisfied. Here, $Z_2 - C_2$ is the most negative number. It will enter in the basis. It is treated as key column. To find key row, find min $\left(\frac{X_B}{x_2}\right)$

$$= \min\left(\frac{3}{7/5}, \frac{4}{1/5}, \frac{6}{9/5}\right)$$

$$= \min\left(\frac{15}{7}, \frac{20}{1} \frac{30}{9}\right) = 15/7$$

$\boxed{7/5}$ is the key element. Make it unity and other elements of key column zero by applying matrix row transformation. We have the second simplex table.

Second simplex table

C_B	B	C_j X_B	1 x_1	2 x_2	3 x_3	–1 x_4
2	x_2	15/7	–1/7	1	0	0
3	x_3	25/7	3/7	0	1	0
← –1	x_4	15/7	$\boxed{6/7}$	0	0	1
	Z_j	$\frac{90}{7}$	$\frac{1}{7}$	2	3	–1
		$Z_j - C_j$	$\frac{-6}{7}$ ↑	0	0	0

Here, optimality conditions are not further satisfied $Z_1 - C_1$ is the most negative number. It will enter in the basis. It is treated as key column key row will be the third row $\boxed{6/7}$ is the key element. Make it unity and other elements of key column zero. Now prepare the third simplex table.

Third simplex table

C_B	B	C_j X_B	1 x_1	2 x_2	3 x_3	–1 x_4
2	x_2	15/6	0	1	0	1/6
3	x_3	15/6	0	0	1	3/6

Contd.

1	x_4	15/6	1	0	0	7/6
	Z_j	15	1	2	3	3
		$Z_j - C_j$	0	0	0	4

Here, all values of $Z_j - C_j \geq 0$. Hence, optimality. Conditions are satisfied. Hence, optimum six is given by $x_1 = x_2 = x_3 = 15/6$, $x_4 = 0$

and $$\text{Max } Z = 15 \quad \textbf{Ans}$$

Example 21 Solve the following L.P.P. using Big. M. method.

$$\text{Min } Z = x_1 + x_2$$
$$\text{S.T. } 2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

Solution

First convert the problem into maximization form and introducing the surplus variables and artificial variable to convert the problem in standard form.

$$\text{Max } Z = -x_1 - 4x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

$$\text{S.T. } 2x_1 + x_2 - S_1 + A_1 = 4$$
$$x_1 + 7x_2 - S_2 + A_2 = 7$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

The initial solution is given by

$$x_1 = x_2 = 0, A_1 = 4, A_2 = 7.$$

Now prepare the initial simplex table.

Initial simplex table

		C_j	2	4	0	0	$-M$	$-M$
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1	A_2
$-M$	A_1	4	2	1	-1	0	1	0
$\leftarrow -M$	A_2	7	1	$\boxed{7}$	0	-1	0	1
	Z_j	$-11M$	$-3M$	$-8M$	M	M	$-M$	$-M$
		$Z_j - C_j$	$-3M - 2$	$-8M - 4$ $\uparrow$	M	M	0	0

Here, all the values of $Z_j - C_j$ are not positive. Therefore, solution is not optimum. To find optimum solution, select the most negative value of $Z_j - C_j$. Here, $Z_2 - C_2 = -8M - 4$ is the most negative value. It will enter in the basis and corresponding column is treated as key column. Find key row by taking min $\left(\dfrac{X_B}{x_2}\right) = \min\left(\dfrac{4}{1}, \dfrac{7}{7}\right) = \min(4, 1) = 1$.

$\therefore$ Artificial variable A_2 will leave the basis. $\boxed{7}$ is treated as key element.

Make it unity and other element of key column to zero by applying matrix row transformation.

First simplex table

C_B	B	C_j / X_B	2 / x_1	4 / x_2	0 / S_1	0 / S_2	$-M$ / A_1
$\leftarrow -M$	A_1	3	$\boxed{13/7}$	0	-1	1/7	1
4	x_2	1	1/7	1	0	$-1/7$	0
	Z_j	$-3M+4$	$\frac{-13}{7}M+\frac{4}{7}$	4	M	$\frac{M}{7}-\frac{1}{7}$	M
		Z_j-C_j	$\frac{-13}{7}M+\frac{10}{7}$ $\uparrow$	0	M	$\frac{M}{7}-\frac{1}{7}$	$2M$

Here, all values of $Z_j - C_j$ are not positive.

Optimality condition is not satisfied.

Now $\qquad R_1 \to \frac{7}{13} R_1$ and then taking

$R_2 \to R_2 - \frac{1}{7} R_1$, we have second simplex table.

Second simplex table

C_B	B	C_j / X_B	2 / x_1	4 / x_2	0 / S_1	0 / S_2
2	x_1	$\frac{21}{13}$	1	0	$\frac{7}{13}$	$\frac{1}{13}$
4	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	$\frac{1}{13}$
	Z_j	$\frac{82}{13}$	2	4	$\frac{10}{13}$	$\frac{6}{13}$
		Z_j-C_j	0	0	$\frac{10}{13}$	$\frac{6}{13}$

Here all the values of $Z_j - C_j$ are positive and no artificial variable appears in optimum basis. Therefore, the required solution is given by

$$x_1 = \frac{21}{13},\ x_2 = \frac{10}{13},\ \text{and min } Z = \frac{82}{13}$$ **Ans.**

7. $$\text{Min } Z = 12x_1 + 20x_2.$$
$$\text{S.T.} \quad 6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

[**Ans.** Min $Z = 205$, $x_1 = 15$, $x_2 = \frac{5}{4}$]

8. $$\text{Min } Z = 5x + 3y$$
$$\text{S.T.} \quad 2x + 4y \leq 12$$
$$2x + 2y = 10$$
$$5x + 2y \geq 10$$
$$x, y \geq 0$$

[**Ans.** $x = 4$, $y = 1$, Min $Z = 23$]

Two-phase Simplex Method

This method is another method to solve a given L.P.P. involving some artificial variable. In this method solution is obtained in two phases.

Phase-I In this phase, we construct an auxiliary L.P.P. leading to a final simplex table. Various steps are given below:

(i) Assign cost – 1 to each artificial variable and cost 0 to all other variables. Also find a new objective function Z^*.

(ii) Solve the auxiliary L.P.P by simplex method until either following three cases arise.

  (i) Max $Z^* < 0$ and at least one artificial variable appears in the optimum basis at positive level. In this case L.P.P. dose not possess any feasible solution.

  (ii) Max $Z^* = 0$ and at least one artificial variable or no artificial variable appears in optimum basis. In both the cases, we go to the phase II.

Phase-II Use solution of phase 1 as the initial value of phase-II. Assign the actual cost to the variables and zero cost of every artificial variable. Delete the artificial variable column from the table. Apply simplex method to the modified simplex table to find the solution.

Example 22 Use two-phase simplex method to solve

$$\text{Max } Z = 5x_1 + 3x_2$$
$$\text{S.T.} \quad 2x_1 + x_2 \leq 1$$
$$x_1 + 4x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Solution

We convert the given problem in standard form by introducing slack variable, surplus variable and artificial variable. Also assign the cost – 1 to artificial variable and the cost 0 to another variables.

We have

Phase-1

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0S_1 + 0S_2 - 1A_1$$
$$\text{S.T. } 2x_1 + x_2 + S_1 = 1$$
$$x_1 + 4x_2 - S_2 + A_1 = 6$$
$$x_1, x_2, S_1, S_2, A_1 \geq 0$$

The initial basic feasible solution is given by

$$x_1 = x_2 = 0, S_1 = 1, A_1 = 6$$

Now prepare the initial simplex table

		c_j	0	0	0	0	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
← 0	S_1	1	2	$\boxed{1}$	1	0	0
–1	A_1	6	1	4	0	–1	1
	Z_j	–6	–1	–4	0	1	–1
		$Z_j - C_j$	–1	–4 ↑	0	1	0

Here, all values of $Z_j - C_j$ are not positive so choose the most negative value of $Z_j - C_j$, i.e., $Z_2 - C_2$ is the most negative value. It will enter in the basis and treated as key column. Find key row by taking min $\left\{\frac{X_B}{x_2}\right\}$.

$$= \min\left\{\frac{1}{1}, \frac{6}{4}\right\} = 1$$

$\boxed{1}$ is key element. Make other elements of key column zero by applying matrix row transformation. We have the first simplex table.

		C_j	0	0	0	0	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	A_1
0	x_2	1	2	1	1	0	0
–1	A_1	2	–7	0	–4	–1	1
	Z_j	–2	7	0	4	1	–1
		$Z_j - C_j$	7	0	4	1	0

Here all values of $Z_j - C_j \geq 0$. Max $Z^* < 0$ and an artificial variable A_1 appears in the basis at positive level. In this case L.P.P. does not possess any feasible solution.

Example 23 *XYZ* company has two bottling plants. One located at G_1 and the other at J. Each plant produces three drinks A, B and C. The number of bottles produced per day are as follows:

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
–1	A_1	40	3	3	–1	0	0	1	0	0
← –1	A_2	40	3	$\boxed{1}$	0	–1	0	0	1	0
–1	A_3	44	2	5	0	0	–1	0	0	1
	Z_j	–124	–8	–9	1	1	1	–1	–1	–1
		$Z_j - C_j$	–8	–9 ↑	1	1	1	0	0	0

Here all the values of $Z_j - C_j$ are not positive. Select the most negative number, i.e., $Z_2 - C_2$ is the most negative number. It will enter in the basis and treated as key column. Find the key row by taking min $\left\{\frac{X_B}{x_2}\right\} = \min\left\{\frac{40}{3}, \frac{40}{3}, \frac{44}{2}\right\} = \left\{\frac{40}{3}\right\}$.

$\boxed{1}$ is the key element.

Now, make all elements of key column zero by applying matrix row transformation.

First simplex table

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
← –1	A_1	–80	–6	0	–1	$\boxed{3}$	0	1	–3	0
0	x_2	40	3	$\boxed{1}$	0	–1	0	0	1	0
–1	A_3	–156	–13	0	0	5	–1	0	–5	1
	Z_j	236	19	0	1	–8	1	–1	8	–1
		$Z_j - C_j$	19	0	1	–8 ↑	1	0	9	0

Further, all values of $Z_j - C_j$ are not positive. Select the most negative number, i.e., $Z_4 - C_4$ is most negative number. It will enter in the basis.

Second simplex table

		C_j	0	0	0	0	0	–1	–1	–1
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	A_3
0	S_2	–80/3	–2	0	–1/3	1	0	1/3	–1	0
0	x_2	$\frac{40}{3}$	1	1	$\frac{-1}{3}$	0	0	1/3	–2/3	0
← –1	A_3	$\frac{-68}{3}$	–3	0	5/3	0	–1	$\frac{-5}{3}$	4	1
	Z_j	$\frac{68}{3}$	3	0	$\frac{-5}{3}$	0	1	$\frac{-5}{3}$	4	–1
		$Z_j - C_j$	3	0	$\frac{-5}{3}$	0	2	$\frac{-8}{3}$	5	0

$$R_3 \to \frac{3}{5} R_3, \text{ we have}$$

C_B	B	C_j / X_B	0 / x_1	0 / x_2	0 / S_1	0 / S_2	0 / S_3	–1 / A_1	–1 / A_2	–1 / A_3
0	S_2	$-80/3$	-2	0	$\frac{-1}{3}$	1	0	$\frac{1}{3}$	-1	0
0	x_2	$\frac{40}{3}$	1	1	$\frac{-1}{3}$	0	0	$\frac{1}{3}$	$\frac{-2}{3}$	0
0	A_1	$\frac{-68}{5}$	$\frac{-9}{5}$	0	1	0	$\frac{-3}{5}$	1	$\frac{-12}{5}$	$\frac{3}{5}$
	Z_j	0	0	0	0	0	0	0	0	0
		$Z_j - C_j$	0	0	0	0	0	1	1	1

Here, Max $Z^* = 0$ and at least one artificial variable appears in the optimum basis, therefore, we go to the phase–II.

Phase-II Consider the final simplex table of phase–I. Provide the actual cost to the variables also delete the artificial variable column from the table and then solve by simplex method.

C_B	B	C_j / X_B	600 / x_1	400 / x_2	0 / S_1	0 / S_2	0 / S_3
0	S_2	$-80/3$	-2	0	$\frac{-1}{3}$	1	0
400	x_2	$40/3$	1	1	$\frac{-1}{3}$	0	0
0	A_1	$-68/3$	$\frac{-9}{5}$	0	1	0	$\frac{-3}{5}$
	Z_j	$\frac{16000}{3}$	800	400	$\frac{400}{3}$	0	0
		$Z_j - C_j$	200	0	$\frac{400}{3}$	0	0

Here, all the values of $Z_j - C_j \geq 0$ and one artificial variable appears in optimum basis.

$\therefore$ $$\text{Max } Z = \frac{16000}{3}, x_2 = \frac{40}{3}, x_1 = 0 \quad \textbf{Ans.}$$

Example 24 Solve the following L.P.P. by two-phase simplex method.

$$\text{Max } Z = 5x_1 - 4x_2 + 3x_3$$

$$\text{S.T. } 2x_1 + x_2 - 6x_3 = 20$$
$$6x_1 + 5x_2 + 10x_3 \leq 76$$
$$8x_1 - 3x_2 + 6x_3 \leq 50$$
$$x_1, x_2\ x_3 \quad \geq 0$$

Solution

Phase-I By introducing slack variables and artificial variable, convert the problem in to standard form and assign the value –1 to the artificial variable and value 0 to all other variables. Find the modified L.P.P.

$$\text{Max } Z^* = 0x_1 - 0x_2 + 0x_3 + 0S_1 + 0S_2 - 1A_1$$
$$\text{S.T.} \quad 2x_1 + x_2 - 6x_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$
$$x_1, x_2, x_3, S_1, S_2, S_3, A \geq 0$$

Initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0,\ A_1 = 20,\ S_1 = 76,\ S_2 = 50.$$

Now prepare the initial simplex table

		C_j	0	0	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	A_1	S_2	S_2
–1	A_1	20	2	1	–6	1	0	0
0	S_1	76	6	5	10	0	1	0
← 0	S_2	50	$\boxed{8}$	–3	6	0	0	1
	Z_j	–20	–2	–1	–1	1	0	0
		$Z_j - C_j$	–2 ↑	–1	6	0	0	0

Taking $R_3 \to \dfrac{R_3}{8}$, then $R_1 \to R_1 - 2R_3$, $R_2 + R_2 - 6R_3$, we have

		C_j	0	0	0	–1	0	0
C_B	B	X_B	x_1	x_2	x_3	A_1	S_1	S_2
–1	A_1	15/2	0	$\boxed{7/4}$	–15/2	1	0	–1/4
0	S_1	77/2	0	$\frac{29}{4}$	11/2	0	1	–3/4
0	x_1	25/4	1	$\frac{-3}{8}$	3/4	0	0	1/8
	Z_j	–15/2	0	$\frac{-7}{4}$	$\frac{15}{2}$	–1	0	1/4
		$Z_j - C_j$	0	$\frac{-7}{4}$ ↑	$\frac{15}{2}$	0	0	$\frac{1}{4}$

$$6x_1 + x_2 + 6x_3 \geq 12$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** Max $Z = -15$, $x_1 = \frac{3}{2}$, $x_2 = 3$, $x_3 = 0$]

2. Min $Z = 12x_1 + 18x_2 + 15x_3$

S.T. $4x_1 + 8x_2 + 6x_3 \geq 64$
$$3x_1 + 6x_2 + 12x_3 \geq 96$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = 0$, $x_2 = \frac{16}{5}$, $x_3 = \frac{32}{5}$, Min $Z = \frac{768}{5}$]

3. Min $Z = 10x_1 + 6x_2 + 2x_3$

S.T. $-x_1 + x_2 + x_3 \geq 1$
$$3x_1 + x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, Min $Z = 10$ $x_3 = 0$]

4. Min $Z = -2x_1 - x_2$

S.T. $x_1 + x_2 \geq 2$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

[**Ans.** Min $Z = -8$, $x_1 = 4$, $x_2 = 0$]

5. Max $Z = 2x_1 + x_2 + x_3$

S.T. $4x_1 + 6x_2 + 3x_3 \leq 8$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$, Max $Z = \frac{64}{21}$]

6. Max $Z = 5x_1 + 3x_2$

S.T. $x_1 + x_2 = 5$
$$x_1 + 2x_2 \leq 6$$
$$5x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

[**Ans.** Min $Z = 23$, $x_1 = 4$, $x_2 = 1$]

7. Max $Z = x_1 + x_2$

S.T. $x_1 + x_2 \geq 2$
$$x_1 + 3x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

[**Ans.** Max $Z = 3$, $x_1 = 3$, $x_2 = 0$]

In matrix notation the primal and dual problem can be written as follows.

Primal Problem Find the column vector X, which

$$\text{Max } Z_P = CX$$

$$\text{S.T. } AX \leq b \text{ and } X \geq 0$$

Dual Problem Find a column vector W, which

$$\text{Min } Z_D = b'W$$

$$\begin{aligned} \text{S.T.} \qquad & A'W \geq C' \\ & W \geq 0 \end{aligned}$$

where A', b', c' are the transposes of A, b and c.

Theorem: Dual and dual of a given primal is the primal.

Proof: Consider the L.P.P.

Primal

$$\begin{aligned} \text{Max } Z_P &= c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{S.T. } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &\leq b_i \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \end{aligned} \tag{1}$$

and

$$x_1, x_2, x_3 \ldots x_n \geq 0$$

Dual The dual of the above primal (1) is given by

$$\begin{aligned} \text{Min } Z_D &= b_1W_1 + b_2W_2 + \ldots + b_nW_m \\ \text{S.T. } a_{11}W_1 + a_{21}W_2 + \ldots + a_{m1}W_m &\geq c_1 \\ a_{12}W_1 + a_{22}W_2 + \ldots + a_{m2}W_m &\geq c_2 \\ &\vdots \\ a_{1n}W_1 + a_{2n}W_2 + \ldots + a_{mn}W_m &\geq c_m \\ W_1, W_2 \ldots W_m &\geq 0 \end{aligned} \tag{2}$$

Now to write the dual of (2), we first write equation (2) in standard form (1). The dual (2) can be written in standard form as

$$\begin{aligned} \text{Max } (-Z_D) &= -b_1W_1 - b_2W_2 \ldots b_nW_m \\ \text{S.T.} \qquad & -a_{11}W_1 - a_{21}W_2 \ldots a_{m1}\ W_m \leq -c_1 \\ & -a_{12}W_1 - a_{22}W_2 \ldots a_{m2}W_m \leq -c_2 \\ & -a_{1n}W_1 - a_{2n}W_2 \ldots a_{mn}W_m \leq -c_m \\ & W_1, W_2, W_3 \ldots w_m \geq 0 \end{aligned} \tag{3}$$

Dual of Dual Now equation (3) is the form (1). Consider (3) as the primal and its dual is given by

$$
\begin{aligned}
\text{Min } Z_D &= -c_1v_1 - c_2v_2 \dots c_nv_n \\
\text{S.T.} \quad & -a_{11}v_1 - a_{12}v_2 \dots a_{1n}v_n \geq -b_1 \\
& -a_{21}v_1 - a_{22}v_2 \dots -a_{2n}v_m \geq -b_2 \\
& \dots \\
& -a_{m1}v_1 - a_{m2}v_2 \dots a_{mn}v_n \geq b_m \\
& v_1, v_2 \dots v_m \geq 0
\end{aligned} \tag{4}
$$

The above dual can also be written as

$$
\begin{aligned}
\text{Max } Z_D &= c_1v_1 + c_2v_2 + \dots + c_nv_n \\
\text{S.T.} \quad & a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \leq b_1 \\
& a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \leq b_2 \\
& \dots \\
& a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}V_n \leq b_m \\
& v_1, v_2, v_3 \dots v_n \geq 0
\end{aligned} \tag{5}
$$

Equation (5) is identical to (1). Hence, it is proved that dual and dual of a given primal is the primal.

Example 25 Write the dual of the problem

$$
\begin{aligned}
\text{Min } Z &= 3x_1 + x_2 \\
\text{S.T.} \quad & 2x_1 + 3x_2 \geq 2 \\
& x_1 + x_2 \geq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Solution

The given L.P.P. is in the standard primal form. In matrix notation it is written as

$$\text{Min } Z_P = (3, 1)\,[x_1, x_2] = CX$$

$$\text{S.T.} \qquad \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$AX \geq b$$

The dual of a given problem is

$$
\begin{aligned}
\text{Max } Z_D &= b'W \\
\text{S.T.} \quad & A'W \leq c'
\end{aligned}
$$

$$
\therefore \qquad \begin{aligned}
\text{Max } Z_D &= [2, 1]\,[W_1, W_2] \\
&= 2W_1 + W_2
\end{aligned}
$$

$$\text{S.T.} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\text{Min } Z = [8, -12, 13]\,[W_1, W_2, W_3]$$

$$\text{S.T.} \begin{bmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

i.e.,

$$\text{Min } Z_D = 8W_1 - 12W_2 + 13W_3$$

$$\begin{aligned} \text{S.T. } 4W_1 - 8W_2 + 5W_3 &\geq 3 \\ -W_1 - W_2 + 0W_3 &\geq -1 \\ 0W_1 - 3W_2 - 6W_3 &\geq 1 \\ W_1, W_2, W_3 &\geq 0 \end{aligned}$$

Example 27 Find the dual of the following.

$$\begin{aligned} &\text{Min } Z = x_1 + 3x_3 \\ &\text{S.T.} \quad 2x_1 + x_3 \leq 3 \\ &\quad x_1 + 2x_2 + 6x_3 \geq 5 \\ &\quad -x_1 + x_3 + 2x_3 = 2 \\ &\quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

The given problem is not in canonical form. First we make it in canonical form.

$$\begin{aligned} &\text{Min } Z = x_1 + 3x_3 \\ &\text{S.T.} \quad -2x_1 - x_3 \geq -3 \\ &\quad x_1 + 2x_2 + 6x_3 \geq 5 \\ &\quad -x_1 + x_2 + 2x_3 \geq 2 \\ &\quad x_1 - x_2 - 2x_3 \geq -2 \\ &\quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

The above problem can be written in matrix form

$$\text{Min } Z = CX$$

$$\text{S.T. } AX \geq b,\ X \geq 0$$

i.e.,

$$\begin{bmatrix} -2 & 0 & -3 \\ 1 & 2 & 6 \\ 1 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -2 \\ 2 \end{bmatrix}$$

Dual of the above primal can be written as

$$\text{Max } Z = b'W$$

$$\text{S}\cdot\text{T}\cdot\ A'W \leq C'$$

$$\begin{bmatrix} -2 & 1 & 1 & -1 \\ -1 & 2 & -1 & 1 \\ 0 & 6 & -2 & 2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \le \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

i.e.,

$$\text{Max } Z_D = -2W_1 + 5W_2 - 2W_3 + 2W_4$$

$$\begin{aligned} \text{S.T. } -2W_1 + W_2 + W_3 - W_4 &\le 1 \\ -W_1 + 2W_2 - W_3 + W_4 &\le 0 \\ 0W_1 + 6W_2 - 2W_3 + 2W_4 &\le 3 \\ W_1, W_2, W_3 &\ge 0 \quad \textbf{Ans.} \end{aligned}$$

Example 29 Find the dual of the following L.P.P. and solve it.

$$\text{Max } Z = 4x_1 + 2x_2$$

$$\begin{aligned} \text{S.T. } x_1 + x_2 &\ge 3 \\ x_1 - x_2 &\ge 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution

The given problem can be written in matrix notation

$$\text{Max } Z = [4, 2]\ [x_1, x_2] = CX$$

$$\text{S.T. } AX \le b$$

i.e.,

$$\begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

Dual of above primal is given by

$$\text{Min } Z = b'W$$

$$\text{S.T. } A'W \ge C'$$

where A', b' and c' are transposes of A, b and c.

$$\text{Min } Z = [-3, -2]\ [w_1, w_2]$$

$$\text{S.T. } \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \ge \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} -W_1 - W_2 &\ge 4 \\ -W_1 + W_2 &\ge 2 \\ W_1, W_2 &\ge 0 \end{aligned}$$

Hence,

$$\text{Min } Z = -3W_1 - 2W_2$$

$$\text{S.T. } -W_1 - W_2 \ge 4$$

$$-W_1 + W_2 \geq 2$$
$$W_1, W_2 \geq 0$$

Now introducing surplus variables and artificial variables convert the problem in standard form.

$$\text{Max } Z^* = 3W_1 + 2W_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

$$\text{S.T.} \quad -W_1 - W_2 - S_1 + A_1 = 4$$
$$-W_1 + W_2 - S_2 + A_2 = 2$$
$$W_1, W_2, S_1\, S_2, A_1, A_2 \geq 0$$

The initial basic feasible solution is given by

$$W_1 = W_2 = 0, A_1 = 4, A_2 = 2$$

Now prepare the initial simplex table.

		C_j	3	2	0	0	$-M$	$-M$
C_B	B	X_B	W_1	W_2	S_1	S_2	A_2	A_2
$-M$	A_1	4	-1	-1	-1	0	1	0
$\leftarrow -M$	A_2	2	-1	$\boxed{1}$	0	-1	0	1
	Z_j	$-6M$	$2M$	0	M	M	$-M$	$-M$
		$Z_j - C_j$	$2M - 3$	-2 $\uparrow$	M	M	0	0

Here, all values of $Z_j - C_j$ are not positive. Hence, current solution is not optimal. To find optimum solution choose the most negative number, i.e., $Z_2 - C_2$ is most negative. It will enter in basis. Taking $R_1 \to R_1 + R_2$, we have

		C_j	3	2	0	0	$-M$	-1
C_B	B	X_B	W_1	W_2	S_1	S_2	A_1	A_2
$-M$	A_1	6	-2	0	-1	-1	1	–
2	W_2	2	-1	1	0	-1	0	–
	Z_j	$-6M + 4$	$2M - 2$	2	M	$M - 2$	$-M$	–
		$Z_j - C_j$	$2M - 5$	0	M	$M - 2$	0	–

Here, all values of $Z_j - C_j \geq 0$ and one artificial variable appears in the basis at positive level. Thus, the problem has no feasible solution.

Example 30 Solve the following L.P.P. by converting it into its dual.

$$\text{Min } Z = 20x_1 + 10x_2$$

$$\text{S.T.} \quad x_1 + x_2 \geq 10$$
$$3x_1 + 2x_2 \geq 24$$
$$x_1, x_2, \geq 0$$

Solution

The dual of the above L.P.P. can be written as

$$\text{Max } Z_D = 10W_1 + 24W_2$$

$$\begin{aligned} \text{S.T.} \quad & W_1 + 3W_2 \le 20 \\ & W_1 + 2W_2 \le 10 \\ & W_1, W_2 \quad \ge 0 \end{aligned}$$

By using slack variables convert the problem into standard form, we have

$$\text{Max } Z = 10W_1 + 24W_2 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T. } W_1 + 3W_2 + S_1 &= 20 \\ W_1 + 2W_2 + S_2 &= 10 \\ W_1, W_2, S_1, S_2 &\ge 0 \end{aligned}$$

Initial basic feasible solution is given by

$$W_1 = W_2 = 0, S_1 = 20, S_2 = 10$$

Now prepare initial simplex table, we have

C_B	B	C_j X_B	10 W_1	24 W_2	0 S_1	0 S_2
0	S_1	20	1	3	1	0
← 0	S_2	10	1	$\boxed{2}$	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	–10	–24 ↑	0	0

Converting the key element $\boxed{2}$ as unity and then taking $R_1 \to R_1 - 3R_2$, we have

C_B	B	C_j X_B	10 W_1	24 W_2	0 S_1	0 S_2
0	S_1	5	–1/2	0	1	–3/2
24	W_2	5	1/2	1	0	1/2
	Z_j	120	12	24	0	12
		$Z_j - C_j$	2	0	0	12

Here, all values of $Z_j - C_j \ge 0$. Hence, optimum solution exists, i.e.,

$$\text{Min } Z = 120, x_1 = 0, x_2 = 12 \quad \textbf{Ans.}$$

By introducing slack variables, convert the given problem in standard form

$$\text{Max } Z = -3x_1 - x_2 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T.} \quad -x_1 - x_2 + S_1 &= -1 \\ -2x_1 - 3x_2 + S_2 &= -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Now display all the values in initial simplex table.

C_B	B	C_j / X_B	-3 / x_1	-1 / x_2	0 / S_1	0 / S_2
0	S_1	–1	–1	–1	1	0
←0	S_2	–2	–2	$\boxed{-3}$	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	3	1	0	0

Here, all values of $Z_j - C_j \geq 0$ and all $X_{Bi} < 0$. Therefore, the current solution is not an optimum, basic feasible solution.

To find optimum solution, find most negative value of X_{Bi} i.e., –2 is the most negative value of X_{Bi}. Therefore, S_2 will leave the basis. To find the entering variable, find max $\left\{\dfrac{Z_j - C_j}{\text{Second row}}\right\}$

$= \text{Max}\left\{\dfrac{3}{-2}, \dfrac{1}{-3}\right\} = -\dfrac{1}{3}$.

∴ x_2 will enter in the basis.

Now prepare the next simplex table.

C_B	B	C_j / X_B	-3 / x_1	-1 / x_2	0 / S_1	0 / S_2
← 0	S_1	–1/3	–1/3	0	1	$\boxed{-1/3}$
–1	x_2	2/3	2/3	1	0	–1/3
	Z_j	$\frac{-2}{3}$	$\frac{-2}{3}$	–1	0	1/3
		$Z_j - C_j$	7/3	0	0	1/3

Since all value of $Z_j - C_j \geq 0$ $X_{Bi} = -1/3 < 0$.

∴ Current solution is not an optimum solution. Since X_{Bi} is negative therefore, S_1 will leave the basis. Find max $\left\{\dfrac{Z_j - C_j}{\text{First row}}\right\} = \left\{\dfrac{\frac{7}{3}}{-1/3}, \dfrac{\frac{1}{3}}{-1/3}\right\} = -1$

C_B	B	C_j X_B	-3 x_1	-1 x_2	0 S_1	0 S_2
0	S_2	1	1	0	–3	1
–1	x_2	1	1	1	–1	0
	Z_j	–1	–1	–1	1	0
		$Z_j - C_j$	2	0	1	0

Since all value of $Z_j - C_j \geq 0$ and $X_{Bi} \geq 0$. Hence, optimum basis feasible solution exists. Also it is given by

$$\text{Max } Z = -1,\ x_1 = 0 \text{ and } x_2 = 1 \textbf{ Ans.}$$

Example 32 Use dual simplex method to solve the following L.P.P.

$$\text{Max } Z = -2x_1 - 3x_2$$

$$\begin{aligned} \text{S.T.} \quad & x_1 + x_2 \geq 2 \\ & 2x_1 + x_2 \leq 10 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution

The above problem can be written in canonical form

$$\text{Max } Z = -2x_1 - 3x_2$$

$$\begin{aligned} \text{S.T.} \quad & -x_1 - x_2 \leq -2 \\ & 2x_1 + x_2 \leq 10 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing slack variables, convert the problem in standard form

$$\text{Max } Z = -2x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3$$

$$\begin{aligned} \text{S.T.} \quad & -x_1 - x_2 + S_1 = -2 \\ & 2x_1 + x_2 + S_2 = 10 \\ & x_1 + x_2 + S_3 = 8 \\ & x_1, x_2, S_1, S_2, S_3 \geq 0 \end{aligned}$$

Initial basic feasible solution is given by

$$x_1 = x_2 = 0,\ S_1 = -2,\ S_2 = 10,\ S_3 = 8.$$

Display the values in the initial simplex table.

		C_j	-2	-3	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
$\leftarrow 0$	S_1	-2	$\boxed{-1}$	-1	1	0	0
0	S_2	10	2	1	0	1	0
0	S_3	8	1	1	0	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	2 $\uparrow$	3	0	0	0

Here all value of $Z_j - C_j \geq 0$ and $X_{Bi} \leq 0 = -2$. Therefore, current solution is not optimum. S_1 will leave the basis. Now find max $\left\{\frac{2}{-1}, \frac{3}{-1}\right\}$ max $\{-2, -3\} = -2$.

$\therefore$ x_1 is the entering variable and corresponding column is treated as key column.

		C_j	-2	-3	0	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2	S_3
-2	x_1	2	1	1	-1	0	0
0	S_2	6	0	-1	2	1	0
0	S_3	6	0	0	1	0	1
	Z_j	-4	-2	-2	1	0	0
		$Z_j - C_j$	0	1	1	0	0

Here, all the values of $Z_j - C_j \geq 0$ and all $X_{Bi} \geq 0$. Hence, optimum solution will exist, i.e.,

$$x_1 = 2, x_2 = 0, \text{Max } Z = -4 \textbf{ Ans.}$$

Example 33 Use dual simplex method to solve the L.P.P.

$$\text{Max } Z = -2x_1 - x_3$$

$$\begin{aligned} \text{S.T.} \quad & x_1 + x_2 - x_3 \geq 5 \\ & x_1 - 2x_2 + 4x_3 \geq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

The above problem can be written in canonical form, i.e.,

$$\text{Max } Z = -2x_1 - 0x_2 - x_3$$

$$\begin{aligned} \text{S.T.} \quad & -x_1 - x_2 + x_3 \leq -5 \\ & -x_1 + 2x_2 - 4x_3 \leq -8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

By introducing slack variables, convert the problem in standard form

$$\text{Max } Z = -2x_1 - 0x_2 - x_3 + 0S_1 + 0S_2$$

$$\begin{aligned}\text{S.T.} \quad & -x_1 - x_2 + x_3 + S_1 = -5 \\ & -x_1 + 2x_2 - 4x_3 + S_2 = -8 \\ & x_1, x_2, x_3, S_1, S_2 \geq 0\end{aligned}$$

Now display the above values in initial simplex table.

C_B	B	C_j / X_B	-2 / x_1	0 / x_2	-1 / x_3	0 / S_1	0 / S_2
0	S_1	-5	-1	-1	1	1	0
$\leftarrow$ 0	S_2	-8	-1	-2	$\boxed{-4}$	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	2	0	1 $\uparrow$	0	0

Here all values of $Z_j - C_j \geq 0$ and all $X_{Bi} < 0$. Therefore, solution is not optimum. Here -8 is the most negative value. Hence, S_2 will leave the basis. Now find max $\left\{\frac{2}{-1}, \frac{1}{-4}\right\} = -\frac{1}{4}$

$\therefore$ x_3 will enter in the basis.

C_B	B	C_j / X_B	-2 / x_1	0 / x_2	-1 / x_3	0 / S_1	0 / S_2
$\leftarrow$ 0	S_1	-7	$-5/4$	$\boxed{-1/2}$	0	1	$\frac{1}{4}$
-1	x_3	2	1/4	$-1/2$	1	0	$-1/4$
	Z_j	-2	$-\frac{1}{4}$	$\frac{1}{2}$	-1	0	$\frac{1}{4}$
		$Z_j - C_j$	$\frac{7}{4}$	$\frac{1}{2}$ $\uparrow$	0	0	$\frac{1}{4}$

Further all the values of $Z_j - C_j \geq 0$ and $X_{Bi} < 0$. Therefore, solution is not optimum. S_1 will leave the basis. Hence, the next simplex table is given by

C_B	B	C_j / X_B	-2 / x_1	0 / x_2	-1 / x_3	0 / S_1	0 / S_2
0	x_2	14	5/2	1	0	-2	$-1/2$
-1	x_3	9	3/2	0	1	-1	$-1/2$
	Z_j	-9	$-3/2$	0	-1	1	1/2
		$Z_j - C_j$	$\frac{1}{2}$	0	0	1	1

$$-4x_1 - x_2 + x_3 \leq -10$$
$$x_1, x_2, x_3 \geq 0$$

[**Ans.** Min $Z = 4W_1 + 15W_2 - 8W_3 + 10W_4 - 10W_5$
S.T. $W_1 + 12W_2 - W_3 + 4W_4 - 4W_5 \geq 20$
$0W_1 + 18W_2 - W_3 + W_4 - W_5 \geq 30$
$-W_1 + 0W_2 - W_3 - W_4 - W_5 \geq 10$
$W_1, W_2, W_3, W_4, W_5 \geq 0$]

5. Max $Z = 2x_1 + 5x_2 + 6x_3$

S.T. $x_1 + 6x_2 - x_3 \leq 3$
$-2x_1 + x_2 + 4x_3 \leq 4$
$x_1 - 5x_2 + 3x_3 \leq 1$
$-3x_1 - 3x_2 + 7x_3 \leq 6$
$x_1, x_2, x_3 \geq 0$

[**Ans.** Min $Z = 3W_1 + 4W_2 + W_3 + 6W_4$
S.T. $W_1 - 2W_2 + W_3 - 3W_4 \geq 2$
$6W_1 + W_2 - 5W_3 - 3W_4 \geq 5$
$-W_1 + 4W_2 + 3W_3 + 7W_4 \geq 6$
$W_1, W_2, W_3, W_4 \geq 0$]

Use duality to solve the L.P.P.

(i) Min $Z = 4x_1 + 2x_2 + 3x_3$

S.T. $2x_1 + 4x_3 \geq 5$
$2x_1 + 3x_2 + x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$

[**Ans.** Min $Z_D = \frac{67}{12}, W_1 = \frac{7}{2}, W_2 = \frac{2}{3}$]

(ii) Max $Z = 3x_1 + 4x_2$

S.T. $x_1 - x_2 \leq 1$
$x_1 + x_2 \geq 4$
$x_1 - 3x_2 \leq 3$
$x_1, x_2 \geq 0$

[No feasible solution exist for dual problem]

(iii) Max $Z = 5x_1 + 12x_2 + 4x_3$

S.T. $x_1 + 2x_2 + x_3 \leq 5$
$2x_1 - x_2 + 3x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$

[Min $Z_D = \frac{141}{5}, W_1 = \frac{29}{5}, W_2 = \frac{-2}{5}$]

(iv) $\text{Min } Z = 2x_2 + 5x_3$

S.T. $x_1 + x_3 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

[Max $Z = 27$, $x_2 = 1$, $x_3 = 5$, $x_2 = 0$]

Use dual simplex method to solve the L.P.P.

(i) $\text{Max } Z = -3x_1 - x_2$

S.T. $x_1 + x_2 \geq 1$

$x_1 + 3x_2 \geq 2$

$x_1, x_2, \geq 0$

[Max $Z = -1$, $x_1 = 0$, $x_2 = 1$]

(ii) $\text{Min } Z = 10x_1 + 6x_2 + 2x_3$

S.T. $-x_1 + x_2 + x_3 \geq 1$

$3x_1 + x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

[Min $Z = 10$, $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$]

(iii) $\text{Min } Z = 30x_1 + 25x_2$

S.T. $2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

[Min $Z = 512.50$, $x_1 = 15$, $x_2 = \frac{5}{2}$]

(iv) $\text{Max } Z = x_1 + x_2$

S.T. $x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_1, x_2 \geq 0$

[Max $Z = 3$, $x_1 = 3$, $x_2 = 0$]

(v) $\text{Max } Z = 10x_1 + 20x_2$

S.T. $2x_1 + 4x_2 \geq 16$

$x_1 + 5x_2 \geq 15$

$x_1, x_2 \geq 0$

[Max Z = Unbounded solution]

(vi) $\text{Min } Z = 2x_1 + 2x_2 + 4x_3$

S.T. $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

[Min $Z = \frac{4}{3}$, $x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 0$]

$$4x_1 + 4x_2 + S_2 = 40$$
$$x_1, x_2, S_1, S_2 \geq 0$$

Initial basis solution is given by

$$x_1 = x_2 = 0, S_1 = 60, S_2 = 40$$

Now form a simplex table.

C_B	B	C_j / X_B	6 / x_1	8 / x_2	0 / S_1	0 / S_2
$\leftarrow 0$	S_1	60	5	$\boxed{10}$	1	0
0	S_2	40	4	4	0	1
	Z_j	0	0	0	0	0
		$Z_j - C_j$	-6	-8 $\uparrow$	0	0

Taking $R_2 \to R_2 - 4R_1$, we have

C_B	B	C_j / X_B	6 / x_1	8 / x_2	0 / S_1	0 / S_2
8	x_2	6	$\frac{1}{2}$	1	$\frac{1}{10}$	0
$\leftarrow 0$	S_2	16	$\boxed{2}$	0	$\frac{-2}{5}$	1
	Z_j	48	4	8	4/5	0
		$Z_j - C_j$	-2 $\uparrow$	0	4/5	0

Now we have the next simplex table.

C_B	B	C_j / X_B	6 / x_1	8 / x_2	0 / S_1	0 / S_2
8	x_2	2	0	1	1/5	$-1/4$
6	x_1	8	1	0	$-1/5$	1/2
	Z_j	64	6	8	2/5	1
		$Z_j - C_j$	0	0	2/5	1

Here $Z_j - C_j \geq 0$ optimum solution exist. Max $Z = 64$, $x_1 = 8$, $x_2 = 2$.

(a) The revised right-hand side constants after incorporating the changes in the constraints are obtained by using the formula.

$$\begin{bmatrix} \text{Basic variable} \\ \text{in the above} \\ \text{optimum table} \end{bmatrix} = \begin{bmatrix} \text{Technological coeff.} \\ \text{columns in the optimal} \\ \text{table w.r.t the basic} \\ \text{variables in intial table} \end{bmatrix} \begin{bmatrix} \text{New R.H.S.} \\ \text{constants} \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

From the above it is clear that $x_1 = 2, x_2 = 3$. Since these values are non-negative, therefore, revised solution is feasible and optimum. The corresponding optimal solution is 36.

(b) We have

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{5} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 20 \\ 40 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \end{bmatrix}$$

$$\Rightarrow x_1 = 16, x_2 = -6.$$

Here $x_2 = -6$ (negative), the solution is infeasible. It can be removed by using dual simplex method, we get

$$x_1 = 4, x_2 = 0, S_1 = 0, S_2 = 24.$$

Hence, $\qquad \text{Max } Z = 24$ **Ans.**

(ii) Making Changes in Objective Function Coefficients

The cost coefficient of objective function undergoes changes over a period of time. Under such situation we can obtain the revised optimum solution from the optimum table of original problem. Also it will be interested to know the range of the coefficient of variable in the objective function over which the optimality is unaffected.

Example 35 Solve the following problem

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3$$

$$\text{S.T.} \qquad \begin{aligned} 2x_1 + 4x_2 + 6x_3 &\leq 24 \\ 3x_1 + 9x_2 + 6x_3 &\leq 30 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Find the range of the objective function coefficients c_1 of the variable x_1 such that optimality is unaffected.

(b) Find the range of objective function coefficients c_2 of the variable x_2 such that the optimality is unaffected.

(c) Check whether optimality is affected, if profit coefficients are changed from (10, 15, 20) to (7, 14, 15). If so, find the revized optimum solution.

Solution

By introducing slack variables convert the problem in standard form

$$\text{Max } Z = 10x_1 + 15x_2 + 20x_3 + 0S_1 + 0S_2$$

$$\begin{aligned} \text{S.T. } 2x_1 + 4x_2 + 6x_3 + S_1 &= 24 \\ 3x_1 + 9x_2 + 6x_2 + S_2 &= 30 \\ x_1, x_2, x_3, S_1, S_2 &\geq 0 \end{aligned}$$

The initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0, S_1 = 24, S_2 = 30$$

Now form the initial simplex table.

		C_j	10	15	20	0	0
C_B	B	X_B	x_1	x_2	x_3	S_1	S_2
← 0	S_1	24	2	4	$\boxed{6}$	1	0
0	S_2	30	3	9	6	0	1
	Z_j	0	0	0	0	0	0
		$Z_j - C_j$	–10	–15	–20 ↑	0	0
20	x_3	4	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{6}$	0
0	S_2	6	$\boxed{1}$	5	0	–1	1
	Z_j	80	$\frac{20}{3}$	$\frac{40}{3}$	20	$\frac{20}{6}$	0
		$Z_j - C_j$	$\frac{-10}{3}$ ↑	$\frac{-5}{3}$	0	0	0
20	x_3	2	0	–1	1	$\frac{1}{2}$	$-\frac{1}{3}$
10	x_1	6	1	5	0	–1	1
	Z_j	100	10	30	20	0	10/3
		$Z_j - C_j$	0	15	0	0	10/3

Here all the values of $Z_j - C_j \geq 0$. Hence, optimum solution is given by Max $Z = 100$, $x_1 = 6$, $x_2 = 0$, $x_3 = 2$.

$$Z_2 - C_2 = 14 + [15, 7] \begin{bmatrix} -1 \\ 5 \end{bmatrix} = 6$$

$$Z_3 - C_3 = 15 + [15, 7] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$Z_4 - C_4 = 0 + [15, 7] \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} = \frac{1}{2}$$

$$Z_5 - C_5 = 0 + [15, 7] \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = 2$$

Since all $Z_j - C_j \geq 0$ the optimality is unaffected.

(iii) Adding a New Constraint

Sometimes a new constraint may be added to an existing L.P.P. as per changing the realities. Under this situation each of the basic variable in new constraint is substituted with the corresponding expression based on the current optimum table. This will give the modified version of the new constraint in terms of only the current non-basic variables.

If the new constraint is satisfied by the values of the current basic variables the constraint is said to be redundant one. Therefore, optimality of the problem is not affected even after including new constraint into the existing problem.

If the new constraint is not satisfied by the values of the current basic variables the optimality of the problem will be affected. Therefore, modified version of the new constraint is to be augmented to the optimal table of the problem and iterated till the optimality is reached.

Example 36 Solve the problem

$$\text{Max } Z = 6x_1 + 8x_2$$

$$\begin{aligned} \text{S.T.} \quad & 5x_1 + 10x_2 \leq 60 \\ & 4x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Check whether the addition of constraint $7x_1 + 2x_2 \leq 65$ affects the optimality. If it does, find the new optimum solution.

(b) Check whether the addition of the constraint $6x_1 + 3x_2 \leq 48$ affects the optimality. If it does, find the new solution.

Form simplex method, the optimum simplex table is given by

C_B	B	C_j X_B	6 x_1	8 x_2	0 S_1	0 S_2
8	x_2	2	0	1	$\frac{1}{5}$	$-\frac{1}{4}$
6	x_1	8	1	0	$\frac{-1}{5}$	$\frac{1}{2}$
	Z_j	64	6	8	2/5	1
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1

(a) The new constraint is given, i.e.,

$$7x_1 + 2x_2 \leq 65$$

This is satisfied by the values of current basic variables ($x_1 = 8$, $x_2 = 2$). Optimality will not be affected even after including the new constraint into the existing L.P.P.

(b) The new constraint is

$$6x_1 + 3x_2 \leq 48$$

This is not satisfied by the values of the current basic variables ($x_1 = 8$, $x_2 = 2$). So the modified form of the new constraint in terms of only non-basic variables is obtained. The standard form of new constraint after including slack variable S_3 is as follows.

$$6x_1 + 3x_2 + S_3 = 48$$

From the above table, we have

$$x_2 + \frac{1}{5} S_1 - \frac{1}{4} S_2 = 2$$

$$x_1 - \frac{1}{5} S_1 + \frac{1}{2} S_2 = 8$$

From the above equations, we have

$$\frac{3}{5} S_1 - \frac{9}{4} S_2 + S_3 = -6$$

Now we have revised table

C_B	B	C_j X_B	6 x_1	8 x_2	0 S_1	0 S_2	0 S_3
8	x_2	2	0	1	1/5	–1/4	0
6	x_1	8	1	0	–1/5	1/2	0
0	S_3	–6	0	0	3/5	–9/4	1
	Z_j	64	6	8	2/5	1	0
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1	0

Solution

Solve the problem by general simplex method. We have optimal simplex table.

		C_j	6	8	0	0
C_B	B	X_B	x_1	x_2	S_1	S_2
8	x_2	2	0	1	1/5	–1/4
6	x_1	8	1	0	–1/5	1/2
	Z_j	64	6	8	$\frac{2}{5}$	1
		$Z_j - C_j$	0	0	$\frac{2}{5}$	1

Here, all the values of $Z_j - C_j \geq 0$. Hence, optimum solution will exist. $x_1 = 8$, $x_2 = 2$, Max $Z = 64$.

(a) Determination of $Z_3 - C_3$. The relative contribution of the new product P_3 is computed by the following formula.

$$Z_j - C_j = C_j - [C_B]\begin{bmatrix}\text{Technical coefficient}\\ \text{of optimal table w.r.t.}\\ \text{the basic variable}\end{bmatrix} \times \begin{bmatrix}\text{Constraint}\\ \text{coefficients of}\\ \text{new variable}\end{bmatrix}$$

$$= 20 - [8, 6]\begin{bmatrix}\frac{1}{5} & -\frac{1}{4}\\ -\frac{1}{5} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}6\\5\end{bmatrix} = \frac{63}{5}$$

Since the value $Z_3 - C_3$ is greater than zero. The solution is not optimal. It means that the inclusion of new product (new variable) in original problem changes the optimality.

(b) Optimization of the modified problem. The constraint coefficients of the new variable X_3 are determined using the following formula.

$$\begin{bmatrix}\text{Revised constraint}\\ \text{coefficient of the}\\ \text{new variable}\end{bmatrix} = \begin{bmatrix}\text{Technical coefficient}\\ \text{of optimal table w.r.t.}\\ \text{the basic variable}\end{bmatrix} \times \begin{bmatrix}\text{Constraint coefficients}\\ \text{of new variable}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{5} & -\frac{1}{4}\\ -\frac{1}{5} & \frac{1}{2}\end{bmatrix} \times \begin{bmatrix}6\\5\end{bmatrix} = \begin{bmatrix}-\frac{1}{20}\\ \frac{13}{10}\end{bmatrix}$$

4. Solve the following L.P.P. by using simplex method.

$$\text{Max } Z = 20x_1 + 80x_2$$

$$\begin{aligned} \text{S.T.} \qquad & 4x_1 + 6x_2 \leq 90 \\ & 8x_1 + 6x_2 \leq 100 \\ & x_1, x_2 \geq 0 \end{aligned}$$

If the new constraint $5x_1 + 4x_2 \leq 80$ is added to this L.P.P. Find the solution to the new problem.

PROJECT NETWORK

LESSON OUTLINE

- The key concepts
- Construction of project network diagram

LEARNING OBJECTIVES

After reading this lesson you should be able to

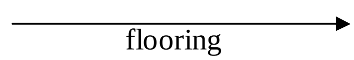
- understand the definitions of important terms
- understand the development of project network diagram
- work out numerical problems

KEY CONCEPTS

Certain key concepts pertaining to a project network are described below:

1. Activity

An activity means a work. A project consists of several activities. An activity takes time. It is represented by an arrow in a diagram of the network. For example, an activity in house construction can be flooring. This is represented as follows:



Construction of a house involves various activities. Flooring is an activity in this project. We can say that a project is completed only when all the activities in the project are completed.

2. Event

It is the beginning or the end of an activity. Events are represented by circles in a project network diagram. The events in a network are called the **nodes**. **Example:**

Starting a punching machine is an activity. Stopping the punching machine is another activity.

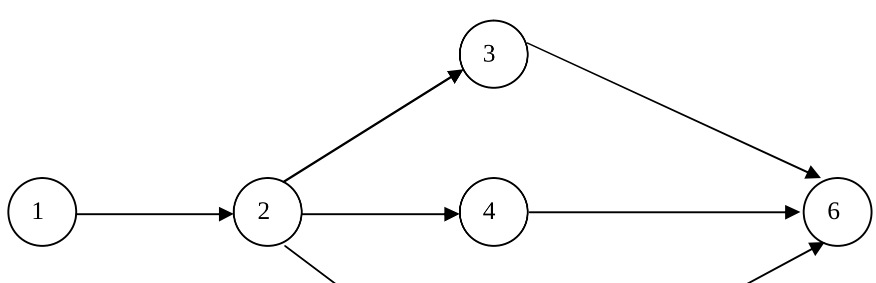
3. Predecessor Event

The event just before another event is called the predecessor event.

4. Successor Event

The event just following another event is called the successor event.

Example: Consider the following.



In this diagram, event 1 is predecessor for the event 2.

Event 2 is successor to event 1.

Event 2 is predecessor for the events 3, 4 and 5.

Event 4 is predecessor for the event 6.

Event 6 is successor to events 3, 4 and 5.

5. Network

A network is a series of related activities and events which result in an end product or service. The activities shall follow a prescribed sequence. For example, while constructing a house, laying the foundation should take place before the construction of walls. Fitting water tapes will be done towards the completion of the construction. Such a sequence cannot be altered.

6. Dummy Activity

A dummy activity is an activity which does not consume any time. Sometimes, it may be necessary to introduce a dummy activity in order to provide connectivity to a network or for the preservation of the logical sequence of the nodes and edges.

7. Construction of a Project Network

A project network consists of a finite number of events and activities, by adhering to a certain specified sequence. There shall be a **start event** and an **end event (or stop event)**. All the other events shall be between the start and the end events. The activities shall be marked by directed arrows. An activity takes the project from one event to another event.

An event takes place at a point of time whereas an activity takes place from one point of time to another point of time.

CONSTRUCTION OF PROJECT NETWORK DIAGRAMS

Problem 1:

Construct the network diagram for a project with the following activities**:**

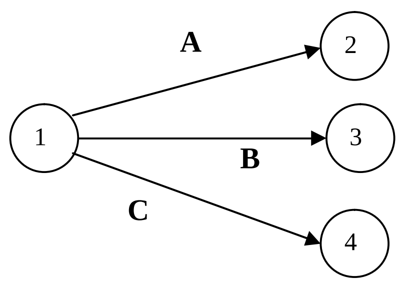
Activity	Name of	Immediate

Event→Event	Activity	Predecessor Activity
1→2	A	-
1→3	B	-
1→4	C	-
2→5	D	A
3→6	E	B
4→6	F	C
5→6	G	D

Solution:

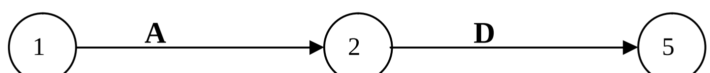
The start event is node 1.

The activities A, B, C start from node 1 and none of them has a predecessor activity. A joins nodes1 and 2; B joins nodes 1 and 3; C joins nodes 1 and 4. So we get the following:

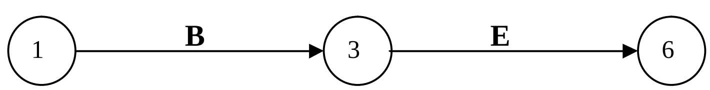


This is a part of the network diagram that is being constructed.

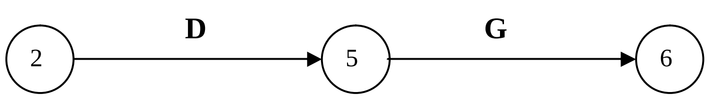
Next, activity D has A as the predecessor activity. D joins nodes 2 and 5. So we get



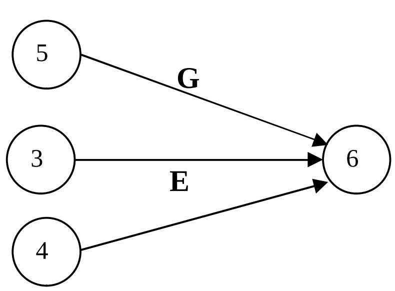
Next, activity E has B as the predecessor activity. E joins nodes 3 and 6. So we get



Next, activity G has D as the predecessor activity. G joins nodes 5 and 6. Thus we obtain



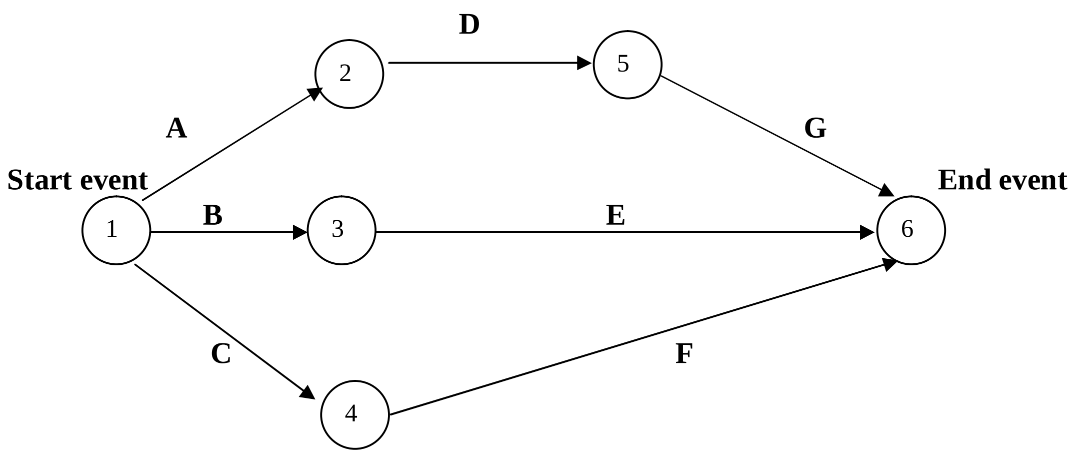
Since activities E, F, G terminate in node 6, we get



F

6 is the end event.

Combining all the pieces together, the following network diagram is obtained for the given project:



We validate the diagram by checking with the given data.

Problem 2:

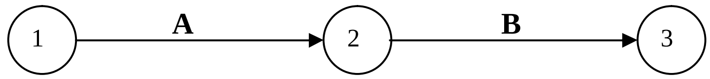
Develop a network diagram for the project specified below:

Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F

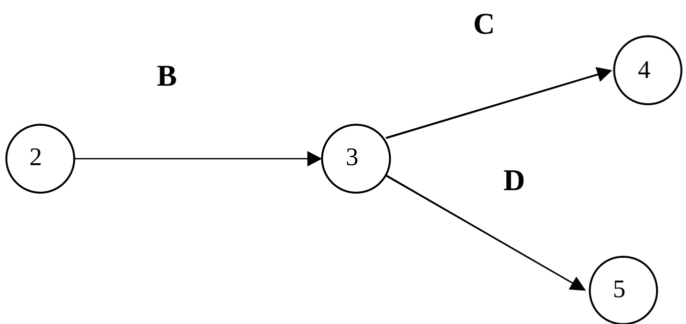
Solution:

Activity A has no predecessor activity. i.e., It is the first activity. Let us suppose that activity A takes the project from event 1 to event 2. Then we have the following representation for A:

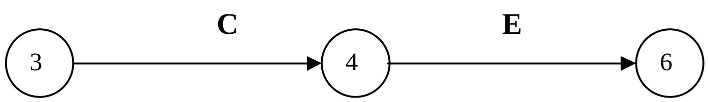
For activity B, the predecessor activity is A. Let us suppose that B joins nodes 2 and 3. Thus we get



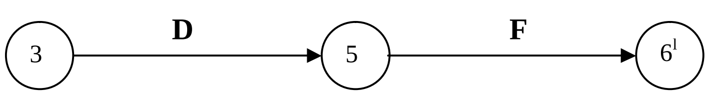
Activities C and D have B as the predecessor activity. Therefore we obtain the following:



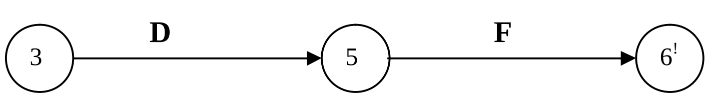
Activity E has D as the predecessor activity. So we get



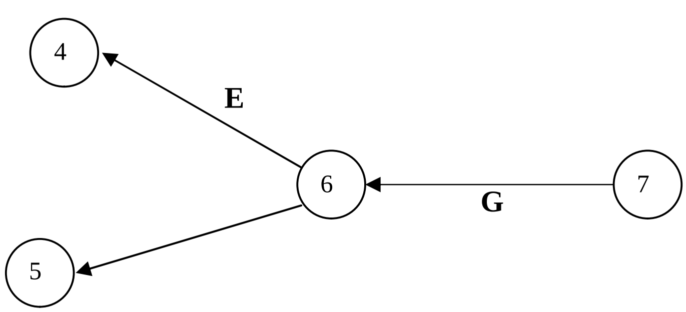
Activity F has D as the predecessor activity. So we get



Activity G has E and F as predecessor activities. This is possible only if nodes 6 and 6^l are one and the same. So, rename node 6^l as node 6. Then we get



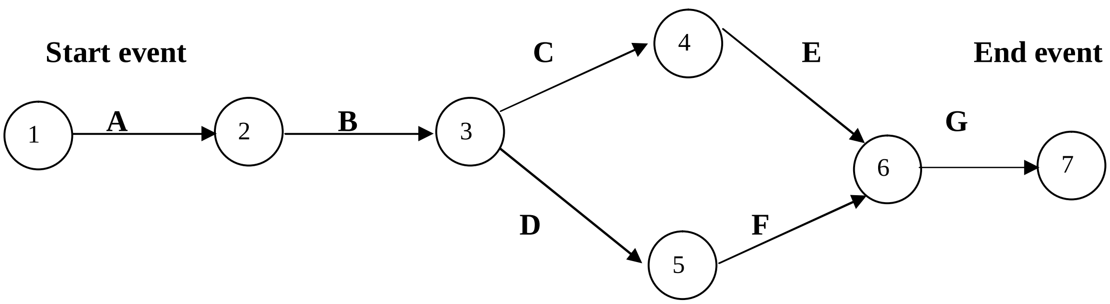
and



F

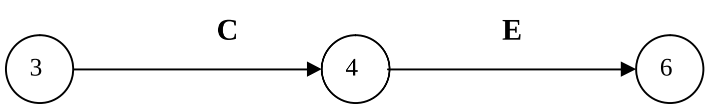
G is the last activity.

Putting all the pieces together, we obtain the following diagram the project network:

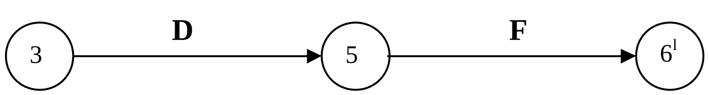


The diagram is validated by referring to the given data.

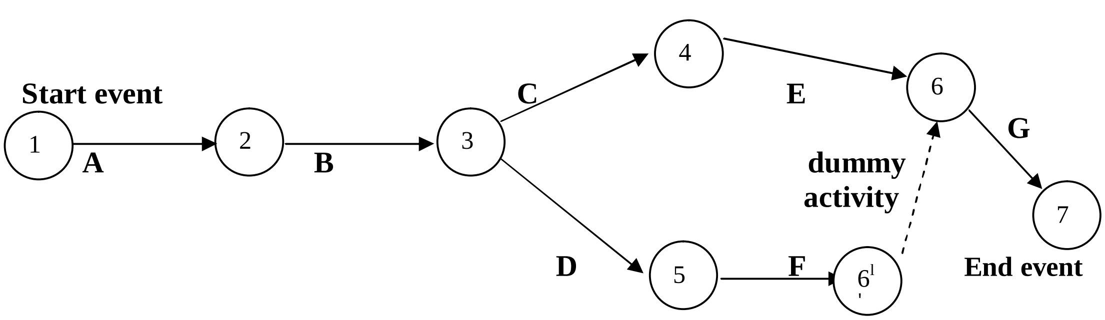
Note: An important point may be observed for the above diagram. Consider the following parts in the diagram



and



We took nodes 6 and 6^l as one and the same. Instead, we can retain them as different nodes. Then, in order to provide connectivity to the network, we join nodes 6^l and 6 by a dummy activity. Then we arrive at the following diagram for the project network:



QUESTIONS:

1. Explain the terms: event, predecessor event, successor event, activity, dummy activity, network.
2. Construct the network diagram for the following project:

Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

CRITICAL PATH METHOD (CPM)

LESSON OUTLINE

- The concepts of critical path and critical activities
- Location of the critical path
- Evaluation of the project completion time

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the definitions of critical path and critical activities
- identify critical path and critical activities
- determine the project completion time

INTRODUCTION

The critical path method (CPM) aims at the determination of the time to complete a project and the important activities on which a manager shall focus attention.

ASSUMPTION FOR CPM

In CPM, it is assumed that precise time estimate is available for each activity.

PROJECT COMPLETION TIME

From the start event to the end event, the time required to complete all the activities of the project in the specified sequence is known as the project completion time.

PATH IN A PROJECT

A continuous sequence, consisting of nodes and activities alternatively, beginning with the start event and stopping at the end event of a network is called a path in the network.

CRITICAL PATH AND CRTICAL ACTIVITIES

Consider all the paths in a project, beginning with the start event and stopping at the end event. For each path, calculate the time of execution, by adding the time for the individual activities in that path.

The path with the largest time is called the **critical path** and the activities along this path are called the **critical activities** or **bottleneck activities**. The activities are called critical because they cannot be delayed. However, a non-critical activity may be delayed to a certain extent. Any delay in a critical activity will delay the completion of the whole project. However, a certain permissible delay in a non –critical activity will not delay the completion of the whole project. It shall be noted that delay in a non-critical activity beyond a limit would certainly delay the completion the whole project. Sometimes, there may be several critical paths for a project. A project manager shall pay special attention to critical activities.

Problem 1:

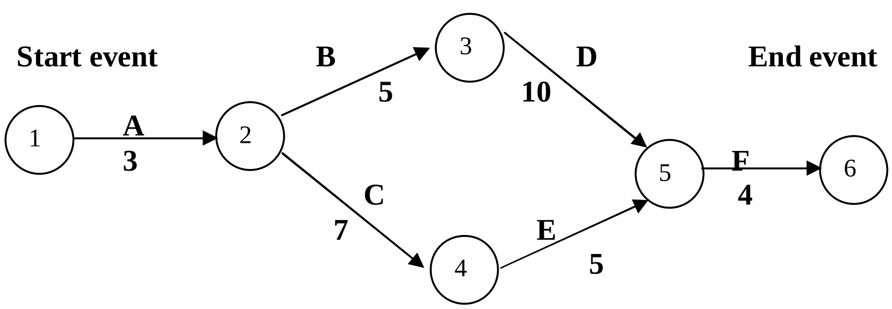
The following details are available regarding a project:

Activity	Predecessor Activity	Duration (Weeks)
A	-	3
B	A	5
C	A	7
D	B	10
E	C	5
F	D,E	4

Determine the critical path, the critical activities and the project completion time.

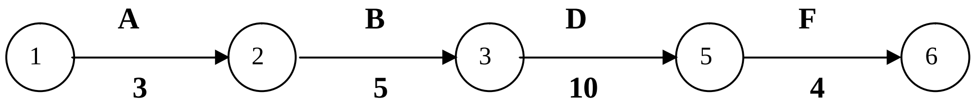
Solution:

First let us construct the network diagram for the given project. We mark the time estimates along the arrows representing the activities. We obtain the following diagram:



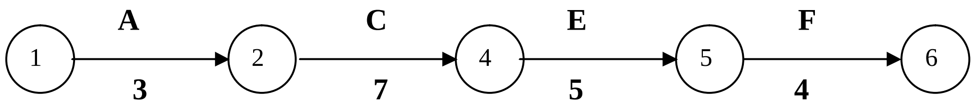
Consider the paths, beginning with the start node and stopping with the end node. There are two such paths for the given project. They are as follows:

Path I



with a time of 3 + 5 + 10 + 4 = 22 weeks.

Path II



with a time of 3 + 7 + 5 + 4 = 19 weeks.

Compare the times for the two paths. Maximum of {22,19} = 22. We see that path I has the maximum time of 22 weeks. Therefore, path I is the critical path. The critical activities are A, B, D and F. The project completion time is 22 weeks.

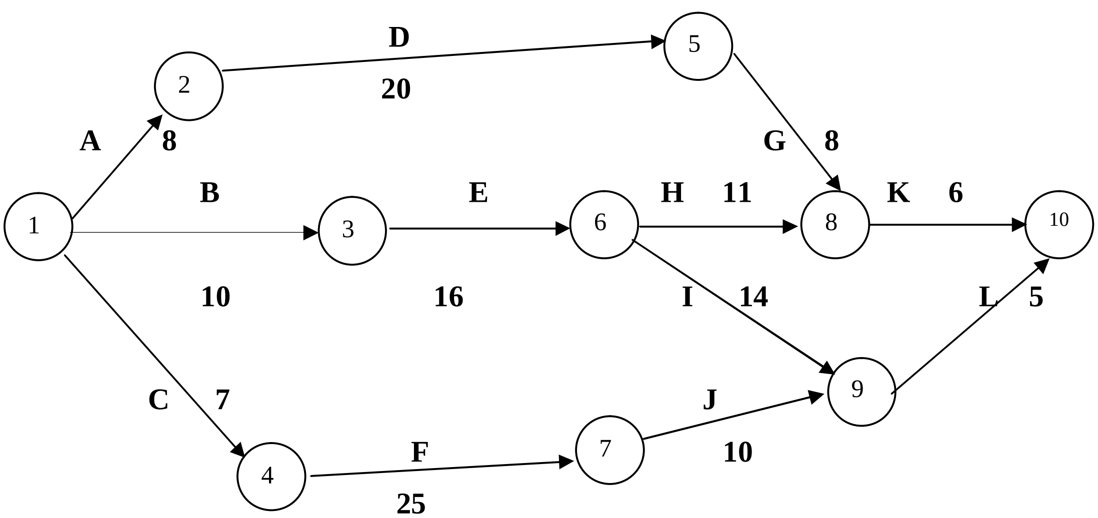
We notice that C and E are non- critical activities.

Time for path I - Time for path II = 22- 19 = 3 weeks.

Therefore, together the non- critical activities can be delayed upto a maximum of 3 weeks, without delaying the completion of the whole project.

Problem 2:

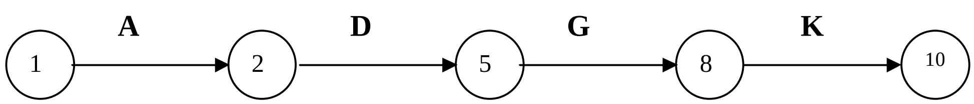
Find out the completion time and the critical activities for the following project:



Solution:

In all, we identify 4 paths, beginning with the start node of 1 and terminating at the end node of 10. They are as follows:

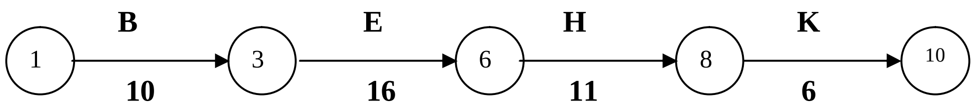
Path I



8 **20** **8** **6**

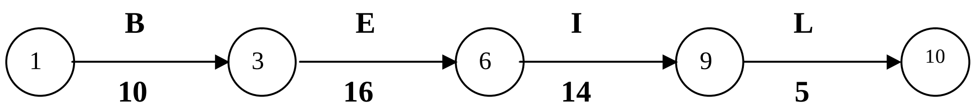
Time for the path = 8 + 20 + 8 + 6 = 42 units of time.

Path II



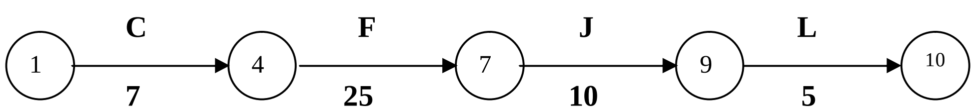
Time for the path = 10 + 16 + 11 + 6 = 43 units of time.

Path III



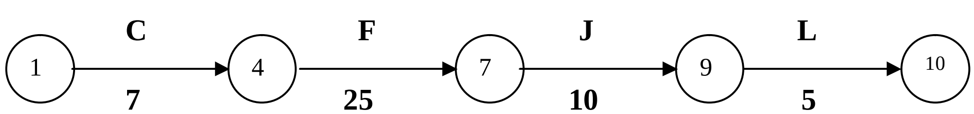
Time for the path = 10 + 16 + 14 + 5 = 45 units of time.

Path IV



Time for the path = 7 + 25 + 10 + 5 = 47 units of time.

Compare the times for the four paths. Maximum of {42, 43, 45, 47} = 47. We see that the following path has the maximum time and so it is the critical path:



The critical activities are C, F, J and L. The non-critical activities are A, B, D, E, G, H, I and K. The project completion time is 47 units of time.

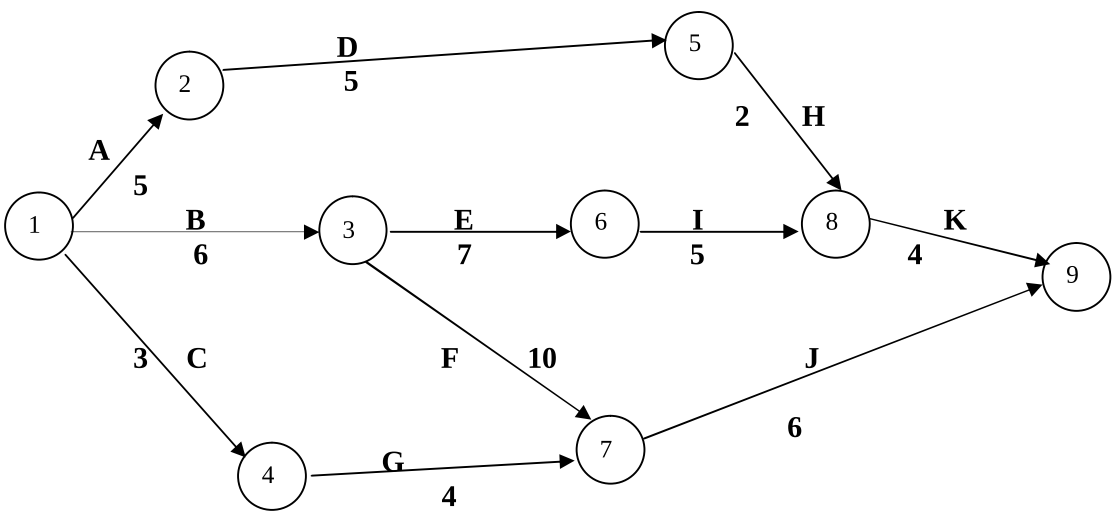
Problem 3:

Draw the network diagram and determine the critical path for the following project:

Activity	Time estimate (Weeks)
1- 2	5
1- 3	6

1- 4	3
2 -5	5
3 -6	7
3 -7	10
4 -7	4
5 -8	2
6 -8	5
7 -9	6
8 -9	4

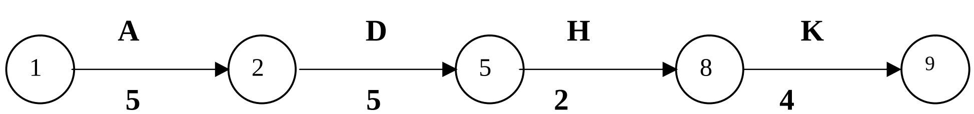
Solution: We have the following network diagram for the project:



Solution:

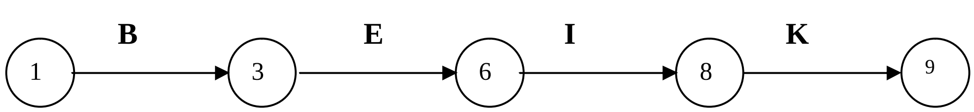
We assert that there are 4 paths, beginning with the start node of 1 and terminating at the end node of 9. They are as follows:

Path I



Time for the path = 5 + 5 + 2 + 4 = 16 weeks.

Path II



6 7 5 4

Time for the path = 6 + 7 + 5 + 4 = 22 weeks.

Path III

$$(1) \xrightarrow[\mathbf{6}]{\mathbf{B}} (3) \xrightarrow[\mathbf{10}]{\mathbf{F}} (7) \xrightarrow[\mathbf{6}]{\mathbf{J}} (9)$$

Time for the path = 6 + 10 + 6 = 16 weeks.

Path IV

$$(1) \xrightarrow[\mathbf{3}]{\mathbf{C}} (4) \xrightarrow[\mathbf{4}]{\mathbf{G}} (7) \xrightarrow[\mathbf{6}]{\mathbf{J}} (9)$$

Time for the path = 3 + 4 + 6 = 13 weeks.

Compare the times for the four paths. Maximum of {16, 22, 16, 13} = 22. We see that the following path has the maximum time and so it is the critical path:

$$(1) \xrightarrow[\mathbf{6}]{\mathbf{B}} (3) \xrightarrow[\mathbf{7}]{\mathbf{E}} (6) \xrightarrow[\mathbf{5}]{\mathbf{I}} (8) \xrightarrow[\mathbf{4}]{\mathbf{K}} (9)$$

The critical activities are B, E, I and K. The non-critical activities are A, C, D, F, G, H and J. The project completion time is 22 weeks.

QUESTIONS:

1. Explain the terms: critical path, critical activities.
2. The following are the time estimates and the precedence relationships of the activities in a project network:

Activity	IMMEDIATE Predecessor Activity	time estimate (weeks)
A	-	4
B	-	7
C	-	3
D	A	6
E	B	4
F	B	7
G	C	6
H	E	10
I	D	3
J	F, G	4
K	H, I	2

Draw the project network diagram. Determine the critical path and the project completion time.

PERT

LESSON OUTLINE

- The concept of PERT
- Estimates of the time of an activity
- Determination of critical path
- Probability estimates

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the importance of PERT
- locate the critical path
- determine the project completion time
- find out the probability of completion of a project before a stipulated time

INTRODUCTION

Programme Evaluation and Review Technique (PERT) is a tool that would help a project manager in project planning and control. It would enable him in continuously monitoring a project and taking corrective measures wherever necessary. This technique involves statistical methods.

ASSUMPTIONS FOR PERT

Note that in CPM, the assumption is that precise time estimate is available for each activity in a project. However, one finds most of the times that this is not practically possible.

In PERT, we assume that it is not possible to have precise time estimate for each activity and instead, probabilistic estimates of time alone are possible. A multiple time estimate approach is followed here. In probabilistic time estimate, the following 3 types of estimate are possible:

1. Pessimistic time estimate (t_p)
2. Optimistic time estimate (t_o)
3. Most likely time estimate (t_m)

The optimistic estimate of time is based on the assumption that an activity will not involve any difficulty during execution and it can be completed within a short period. On the other hand, a pessimistic estimate is made on the assumption that there would be unexpected problems during the execution of an activity and hence it would consume more time. The most likely time estimate is made in between the optimistic and the pessimistic estimates of time. Thus the three estimates of time have the relationship

$t_o \le t_m \le t_p$.

Practically speaking, neither the pessimistic nor the optimistic estimate may hold in reality and it is the most likely time estimate that is expected to prevail in almost all cases. Therefore, it is preferable to give more weight to the most likely time estimate.

We give a weight of 4 to most likely time estimate and a weight of 1 each to the pessimistic and optimistic time estimates. We arrive at a time estimate (t_e) as the weighted average of these estimates as follows:

$$t_e = \frac{t_o + 4\, t_m + t_p}{6}$$

Since we have taken 6 units (1 for t_p, 4 for t_m and 1 for t_o), we divide the sum by 6. With this time estimate, we can determine the project completion time as applicable for CPM.

Since PERT involves the average of three estimates of time for each activity, this method is very practical and the results from PERT will be have a reasonable amount of reliability.

MEASURE OF CERTAINTY

The 3 estimates of time are such that

$$t_o \le t_m \le t_p .$$

Therefore the range for the time estimate is $t_p - t_o$.

The time taken by an activity in a project network follows a distribution with a standard deviation of one sixth of the range, approximately.

i.e., The standard deviation = $\sigma = \dfrac{t_p - t_o}{6}$

and the variance = $\sigma^2 = \left(\dfrac{t_p - t_o}{6}\right)^2$

The certainty of the time estimate of an activity can be analysed with the help of the variance. The greater the variance, the more uncertainty in the time estimate of an activity.

Problem 1:

Two experts A and B examined an activity and arrived at the following time estimates.

Expert	Time Estimate

	t_o	t_m	t_p
A	4	6	8
B	4	7	10

Determine which expert is more certain about his estimates of time:

Solution:

$$\text{Variance } (\sigma^2) \text{ in time estimates} = \left(\frac{t_p - t_o}{6}\right)^2$$

In the case of expert A, the variance = $\left(\frac{8-4}{6}\right)^2 = \frac{4}{9}$

As regards expert B, the variance = $\left(\frac{10-4}{6}\right)^2 = 1$

So, the variance is less in the case of A. Hence, it is concluded that the expert A is more certain about his estimates of time.

Determination of Project Completion Time in PERT

Problem 2:

Find out the time required to complete the following project and the critical activities:

Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	4	6
B	A	3	6	9
C	A	8	10	12
D	B	9	12	15
E	C	8	9	10
F	D, E	16	21	26
G	D, E	19	22	25
H	F	2	5	8
I	G	1	3	5

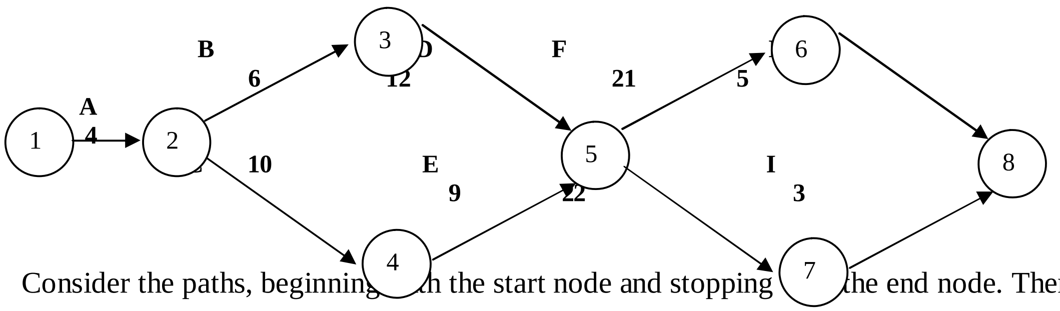
Solution:

From the three time estimates t_p, t_m and t_o, calculate t_e for each activity. We obtain the following table:

Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	$t_o + 4t_m + t_p$	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
A	2	16	6	24	4
B	3	24	9	36	6
C	8	40	12	60	10
D	9	48	15	72	12
E	8	36	10	54	9
F	16	84	26	126	21

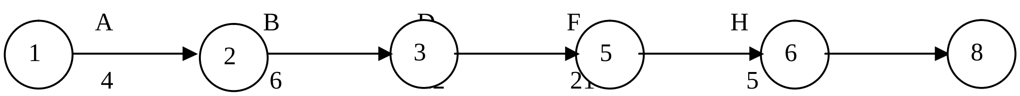
G	19	88	25	132	22
H	2	20	8	30	5
I	1	12	5	18	3

Using the single time estimates of the activities, we get the following network diagram for the project.



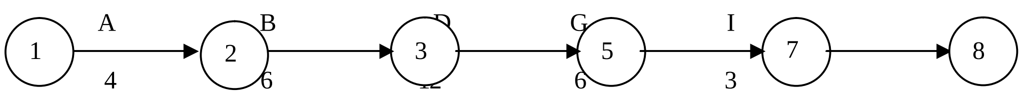
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



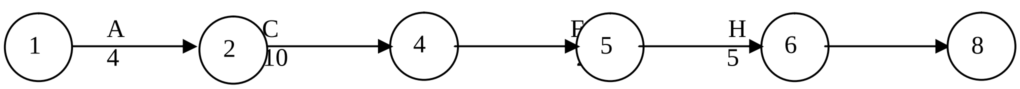
Time for the path: 4+6+12+21+5 = 48 days.

Path II



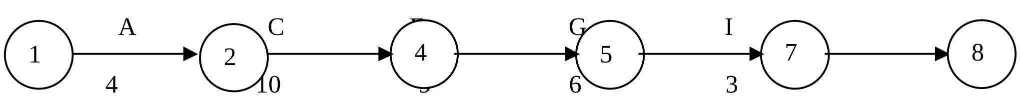
Time for the path: 4+6+12+ 6+3 = 31 days.

Path III



Time for the path: 4+10+9+ 21+5 = 49 days.

Path IV



Time for the path: 4+10+9+ 6+3 = 32 days.

Compare the times for the four paths.
Maximum of {48, 31, 49, 32} = 49.
We see that Path III has the maximum time.
Therefore the critical path is Path III. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 8$.
The critical activities are A, C, E, F and H.
The non-critical activities are B, D, G and I.
Project time (Also called project length) = 49 days.

Problem 3:

Find out the time, variance and standard deviation of the project with the following time estimates in weeks:

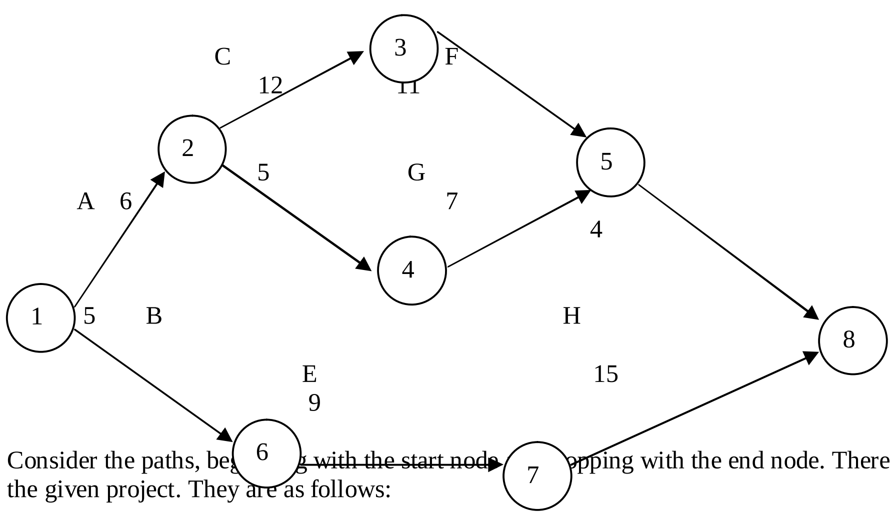
Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)
1-2	3	6	9
1-6	2	5	8
2-3	6	12	18
2-4	4	5	6
3-5	8	11	14
4-5	3	7	11
6-7	3	9	15
5-8	2	4	6
7-8	8	16	18

Solution:

From the three time estimates t_p, t_m and t_o, calculate t_e for each activity. We obtain the following table:

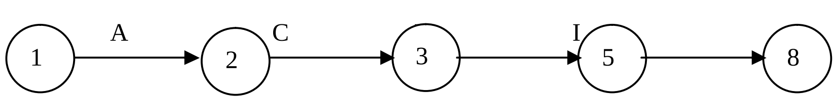
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	t_o+ $4t_m$+ t_p	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
1-2	3	24	9	36	6
1-6	2	20	8	30	5
2-3	6	48	18	72	12
2-4	4	20	6	30	5
3-5	8	44	14	66	11
4-5	3	28	11	42	7
6-7	3	36	15	54	9
5-8	2	16	6	24	4
7-8	8	64	18	90	15

With the single time estimates of the activities, we get the following network diagram for the project.



Consider the paths, beg... g with the start node ... opping with the end node. There are three such paths for the given project. They are as follows:

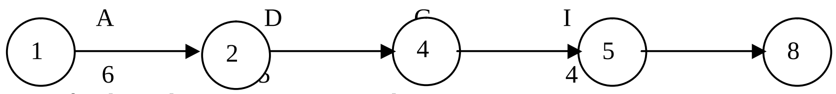
Path I



6 12 11 4

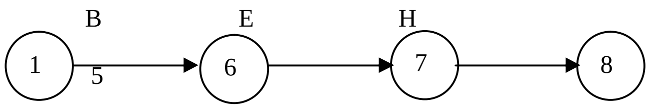
Time for the path: 6+12+11+4 = 33 weeks.

Path II



Time for the path: 6+5+7+ 4= 22 weeks.

Path III



Time for the path: 5+9+15 = 29 weeks.

Compare the times for the three paths.
Maximum of {33, 22, 29} = 33.
It is noticed that Path I has the maximum time.
Therefore the critical path is Path I. i.e., $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 8$
The critical activities are A, C, F and I.
The non-critical activities are B, D, G and H.
Project time = 33 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range (t_p - t_o)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: $1\rightarrow2$	3	6	9	6	1	1
C: $2\rightarrow3$	6	12	18	12	2	4
F: $3\rightarrow5$	8	11	14	6	1	1
I: $5\rightarrow8$	2	4	6	4	2/3	4/9

Variance of project time (Also called Variance of project length) =
Sum of the variances for the critical activities = 1+4+1+ 4/9 = 58/9 Weeks.
Standard deviation of project time = $\sqrt{\text{Variance}}$ = $\sqrt{58/9}$ = 2.54 weeks.

Problem 4

A project consists of seven activities with the following time estimates. Find the probability that the project will be completed in 30 weeks or less.

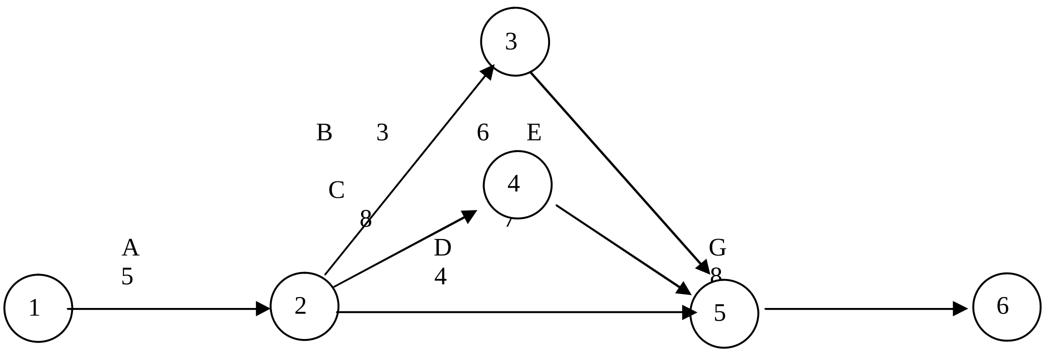
Activity	Predecessor Activity	Optimistic time estimate (t_o days)	Most likely time estimate (t_m days)	Pessimistic time estimate (t_p days)
A	-	2	5	8
B	A	2	3	4
C	A	6	8	10
D	A	2	4	6
E	B	2	6	10
F	C	6	7	8
G	D, E, F	6	8	10

Solution:

From the three time estimates t_p, t_m and t_o, calculate t_e for each activity. The results are furnished in the following table:

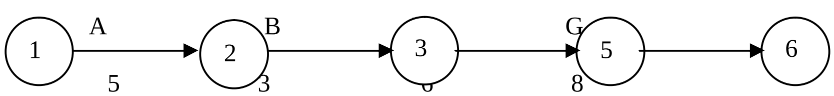
Activity	Optimistic time estimate (t_o)	4 x Most likely time estimate	Pessimistic time estimate (t_p)	t_o+ 4t_m+ t_p	Time estimate $t_e = \frac{t_o + 4\,t_m + t_p}{6}$
A	2	20	8	30	5
B	2	12	4	18	3
C	6	32	10	48	8
D	2	16	6	24	4
E	2	24	10	36	6
F	6	28	8	42	7
G	6	32	10	48	8

With the single time estimates of the activities, the following network diagram is constructed for the project.



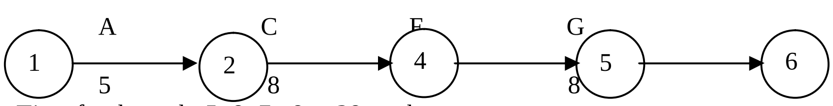
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



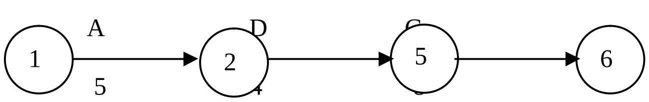
Time for the path: 5+3+6+8 = 22 weeks.

Path II



Time for the path: 5+8+7+ 8 = 28 weeks.

Path III



Time for the path: 5+4+8 = 17 weeks.

Compare the times for the three paths.
Maximum of {22, 28, 17} = 28.
It is noticed that Path II has the maximum time.

Therefore the critical path is Path II. i.e., $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$.
The critical activities are A, C, F and G.
The non-critical activities are B, D and E.
Project time = 28 weeks.

Calculation of Standard Deviation and Variance for the Critical Activities:

Critical Activity	Optimistic time estimate (t_o)	Most likely time estimate (t_m)	Pessimistic time estimate (t_p)	Range ($t_p - t_o$)	Standard deviation = $\sigma = \frac{t_p - t_o}{6}$	Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A: $1 \rightarrow 2$	2	5	8	6	1	1
C: $2 \rightarrow 4$	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$
F: $4 \rightarrow 5$	6	7	8	2	$\frac{1}{3}$	$\frac{1}{9}$
G: $5 \rightarrow 6$	6	8	10	4	$\frac{2}{3}$	$\frac{4}{9}$

Standard deviation of the critical path = $\sqrt{2}$ = 1.414

The standard normal variate is given by the formula

$$Z = \frac{Given\ value\ of\ t - Expected\ value\ of\ t\ \text{in}\ the\ critical\ path}{SD\ for\ the\ critical\ path}$$

So we get $Z = \frac{30 - 28}{1.414}$ = 1.414

We refer to the Normal Probability Distribution Table.
Corresponding to Z = 1.414, we obtain the value of 0.4207
We get 0.5 + 0.4207 = 0. 9207
Therefore the required probability is 0.92
i.e., There is 92% chance that the project will be completed before 30 weeks. In other words, the chance that it will be delayed beyond 30 weeks is 8%

QUESTIONS:

1. Explain how time of an activity is estimated in PERT.
2. Explain the measure of certainty in PERT.
3. The estimates of time in weeks of the activities of a project are as follows:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	A	8	11	20
C	A	10	15	20
D	B	12	18	24

E	C	8	13	24
F	C	4	7	16
G	D,F	14	18	28
H	E	10	12	14
I	G,H	7	10	19

Determine the critical activities and the project completion time.

4. Draw the network diagram for the following project. Determine the time, variance and standard deviation of the project.:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	12	14	22
B	-	16	17	24
C	A	14	15	16
D	A	13	18	23
E	B	16	18	20
F	D,E	13	14	21
G	C,F	6	8	10

5. Consider the following project with the estimates of time in weeks:

Activity	Predecessor Activity	Optimistic estimate of time	Most likely estimate of time	Pessimistic estimate of time
A	-	2	4	6
B	-	3	5	7
C	A	5	6	13
D	A	4	8	12
E	B,C	5	6	13
F	D,E	6	8	14

Find the probability that the project will be completed in 27 weeks.

NORMAL DISTRIBUTION TABLE

Area Under Standard Normal Distribution

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

EARLIEST AND LATEST TIMES

LESSON OUTLINE

- The concepts of earliest and latest times
- The concept of slack
- Numerical problems

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concepts of earliest and latest times
- understand the concept of slack
- calculate the earliest and latest times
- find out the slacks
- identify the critical activities
- carry out numerical problems

INTRODUCTION

A project manager has the responsibility to see that a project is completed by the stipulated date, without delay. Attention is focused on this aspect in what follows.

Key concepts

Certain key concepts are introduced below.

EARLIEST TIMES OF AN ACTIVITY

We can consider (i) Earliest Start Time of an activity and (ii) Earliest Finish Time of an activity.

Earliest Start Time of an activity is the earliest possible time of starting that activity on the condition that all the other activities preceding to it were began at the earliest possible times.

Earliest Finish Time of an activity is the earliest possible time of completing that activity. It is given by the formula.

The Earliest Finish Time of an activity = The Earliest Start Time of the activity + The estimated duration to carry out that activity.

LATEST TIMES OF AN ACTIVITY

We can consider (i) Latest Finish Time of an activity and (ii) Latest Start Time of an activity.

Latest Finish Time of an activity is the latest possible time of completing that activity on the condition that all the other activities succeeding it are carried out as per the plan of the management and without delaying the project beyond the stipulated time.

Latest Start Time of an activity is the latest possible time of beginning that activity. It is given by the formula

Latest Start Time of an activity = The Latest Finish Time of the activity - The estimated duration to carry out that activity.

TOTAL FLOAT OF AN ACTIVITY

Float seeks to measure how much delay is acceptable. It sets up a control limit for delay.

The total float of an activity is the time by which that activity can be delayed without delaying the whole project. It is given by the formula

Total Float of an Activity = Latest Finish Time of the activity - Earliest Finish Time of that activity.

It is also given by the formula

Total Float of an Activity = Latest Start Time of the activity - Earliest Start Time of that activity.

Since a delay in a critical activity will delay the execution of the whole project, the total float of a critical activity must be zero.

EXPECTED TIMES OF AN EVENT

An event occurs at a point of time. We can consider (i) Earliest Expected Time of Occurrence of an event and (ii) Latest Allowable Time of Occurrence an event.

The Earliest Expected Time of Occurrence of an event is the earliest possible time of expecting that event to happen on the condition that all the preceding activities have been completed.

The Latest Allowable Time of Occurrence of an event is the latest possible time of expecting that event to happen without delaying the project beyond the stipulated time.

PROCUDURE TO FIND THE EARLIEST EXPECTED TIME OF AN EVENT

Step 1. Take the Earliest Expected Time of Occurrence of the Start Event as zero.

Step 2. For an event other than the Start Event, find out all paths in the network which connect the Start node with the node representing the event under consideration.

Step 3. In the **"Forward Pass"** (i.e., movement in the network from left to right), find out the sum of the time durations of the activities in each path identified in Step 2.

Step 4. The path with the longest time in Step 3 gives the Earliest Expected Time of Occurrence of the event

Working Rule for finding the earliest expected time of an event:

For an event under consideration, locate all the predecessor events and identify their earliest expected times. With the earliest expected time of each event, add the time duration of the activity connecting that event to the event under consideration. The maximum among all these values gives the Earliest Expected Time of Occurrence of the event.

PROCUDURE TO FIND THE LATEST ALLOWABLE TIME OF AN EVENT

We consider the **"Backward Pass"** (i.e., movement in the network from right to left).

The latest allowable time of occurrence of the End Node must be the time of completion of the project. Therefore it shall be equal to the time of the critical path of the project.

Step 1. Identify the latest allowable time of occurrence of the End Node.

Step 2. For an event other than the End Event, find out all paths in the network which connect the End node with the node representing the event under consideration.

Step 3. In the **"Backward Pass"** (i.e., movement in the network from right to left), subtract the time durations of the activities along each such path.

Step 4. The Latest Allowable Time of Occurrence of the event is determined by the path with the longest time in Step 3. In other words, the smallest value of time obtained in Step 3 gives the Latest Allowable Time of Occurrence of the event.

Working Rule for finding the latest allowable time of an event:

For an event under consideration, locate all the successor events and identify their latest allowable times. From the latest allowable time of each successor event, subtract the time duration of the activity that begins with the event under consideration. The minimum among all these values gives the Latest Allowable Time of Occurrence of the event.

SLACK OF AN EVENT

The allowable time gap for the occurrence of an event is known as the slack of that event. It is given by the formula

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

SLACK OF AN ACTIVITY

The slack of an activity is the float of the activity.

Problem 1:

The following details are available regarding a project:

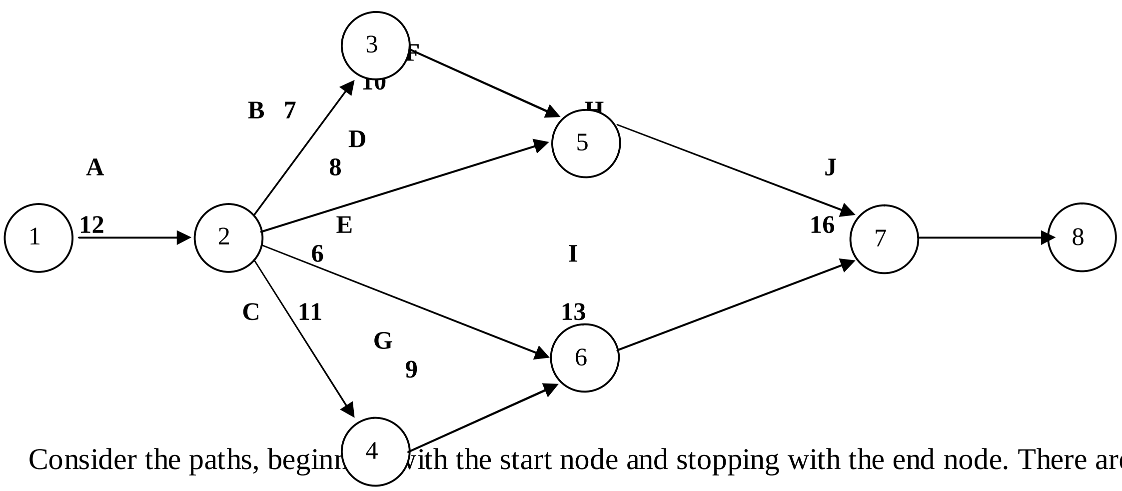
Activity	Predecessor Activity	Duration (Weeks)
A	-	12

B	A	7
C	A	11
D	A	8
E	A	6
F	B	10
G	C	9
H	D, F	14
I	E, G	13
J	H, I	16

Determine the earliest and latest times, the total float for each activity, the critical activities and the project completion time.

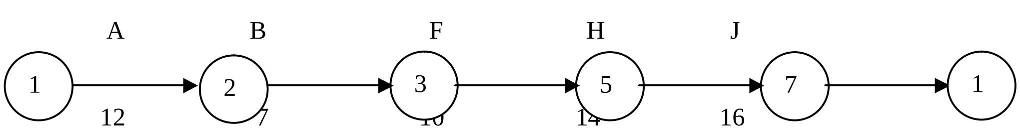
Solution:

With the given data, we construct the following network diagram for the project.



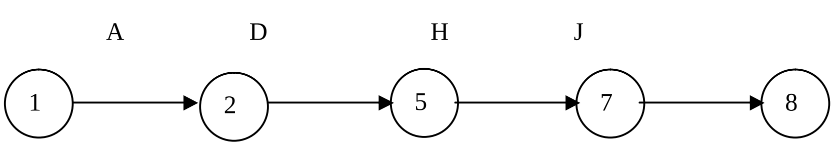
Consider the paths, beginning with the start node and stopping with the end node. There are four such paths for the given project. They are as follows:

Path I



Time of the path = 12 + 7 + 10 + 14 + 16 = 59 weeks.

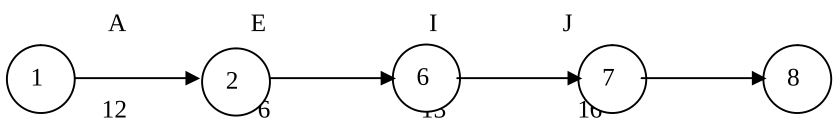
Path II



12 8 14 16

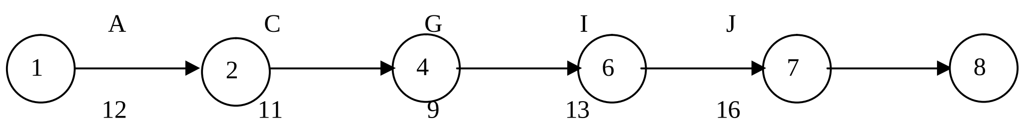
Time of the path = 12 + 8 + 14 + 16 = 50 weeks.

Path III



Time of the path = 12 + 6 + 13 + 16 = 47 weeks.

Path IV



Time of the path = 12 + 11 + 9 + 13 + 16 = 61 weeks.

Compare the times for the four paths. Maximum of {51, 50, 47, 61} = 61. We see that the maximum time of a path is 61 weeks.

Forward pass:

Calculation of Earliest Expected Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 12 = 12
3	Time for Node 2 + Time for Activity B = 12 + 7 = 19
4	Time for Node 2 + Time for Activity C = 12 + 11 = 23
5	Max {Time for Node 2 + Time for Activity D, Time for Node 3 + Time for Activity F} = Max {12 + 8, 19 + 10} = Max {20, 29} = 29
6	Max {Time for Node 2 + Time for Activity E, Time for Node 4 + Time for Activity G} = Max {12 + 6, 23 + 9} = Max {18, 32} = 32
7	Max {Time for Node 5 + Time for Activity H, Time for Node 6 + Time for Activity I} = Max {29 + 14, 32 + 13} = Max {43, 45} = 45
8	Time for Node 7 + Time for Activity J = 45 + 16 = 61

Using the above values, we obtain the Earliest Start Times of the activities as follows:

Activity	Earliest Start Time (Weeks)

A	0
B	12
C	12
D	12
E	12
F	19
G	23
H	29
I	32
J	45

Backward pass:

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 61
7	Time for Node 8 - Time for Activity J = 61 -16 = 45
6	Time for Node 7 - Time for Activity I = 45 -13 = 32
5	Time for Node 7 - Time for Activity H = 45 -14 = 31
4	Time for Node 6 - Time for Activity G = 32 - 9 = 23
3	Time for Node 5 - Time for Activity F = 31- 10 = 21
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C, Time for Node 5 - Time for Activity D, Time for Node 6 - Time for Activity E} = Min {21 - 7, 23 - 11, 31 - 8, 32 - 6} = Min {14, 12, 23, 26} = 12
1	Time for Node 2 - Time for Activity A = 12- 12 = 0

Using the above values, we obtain the Latest Finish Times of the activities as follows:

Activity	Latest Finish Time (Weeks)
J	61
I	45
H	45
G	32
F	31

E	32
D	31
C	23
B	21
A	12

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	12	0	12	0	12	0
B	7	12	19	14	21	2
C	11	12	23	12	23	0
D	8	12	20	23	31	11
E	6	12	18	26	32	14
F	10	19	29	21	31	2
G	9	23	32	23	32	0
H	14	29	43	31	45	2
I	13	32	45	32	45	0
J	16	45	61	45	61	0

The activities with total float = 0 are A, C, G, I and J. They are the critical activities.
Project completion time = 61 weeks.

Problem 2:

The following are the details of the activities in a project:

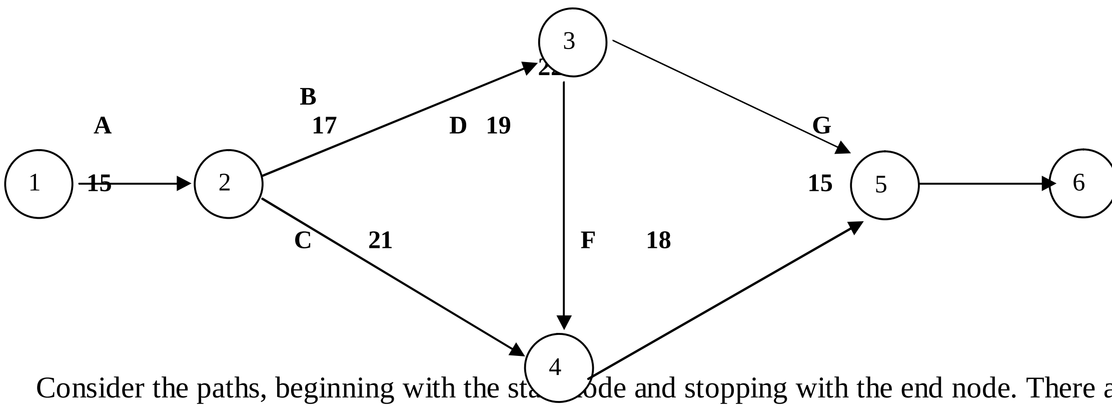
Activity	Predecessor Activity	Duration (Weeks)
A	-	15
B	A	17
C	A	21
D	B	19
E	B	22

F	C, D	18
G	E, F	15

Calculate the earliest and latest times, the total float for each activity and the project completion time.

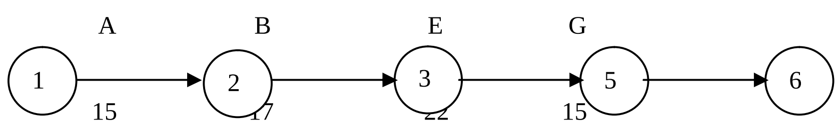
Solution:

The following network diagram is obtained for the given project.



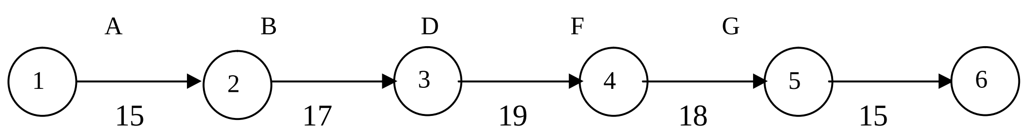
Consider the paths, beginning with the start node and stopping with the end node. There are three such paths for the given project. They are as follows:

Path I



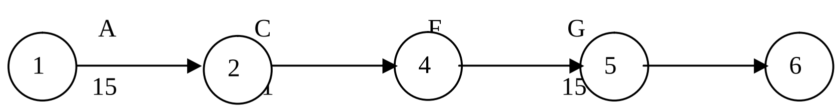
Time of the path = 15 + 17 + 22 + 15 = 69 weeks.

Path II



Time of the path = 15 + 17 + 19 + 18 + 15 = 84 weeks.

Path III



Time of the path = 15 + 21 + 18 + 15 = 69 weeks.

Compare the times for the three paths. Maximum of {69, 84, 69} = 84. We see that the maximum time of a path is 84 weeks.

Forward pass:

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 15 = 15
3	Time for Node 2 + Time for Activity B = 15 + 17 = 32
4	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity D} = Max {15 + 21, 32 + 19} = Max {36, 51} = 51
5	Max {Time for Node 3 + Time for Activity E, Time for Node 4 + Time for Activity F} = Max {32 + 22, 51 + 18} = Max {54, 69} = 69
6	Time for Node 5 + Time for Activity G = 69 + 15 = 84

Calculation of Earliest Time for Activities

Activity	Earliest Start Time (Weeks)
A	0
B	15
C	15
D	32
E	32
F	51
G	69

Backward pass:

Calculation of the Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
6	Maximum time of a path in the network = 84
5	Time for Node 6 - Time for Activity G = 84 -15 = 69
4	Time for Node 5 - Time for Activity F = 69 - 18 = 51
3	Min {Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity E} = Min {51 - 19, 69 - 22} = Min {32, 47} = 32
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity C} = Min {32 - 17, 51 - 21} = Min {15, 30} = 15
1	Time for Node 2 - Time for Activity A = 15 - 15 = 0

Calculation of the Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
G	84

F	69
E	69
D	51
C	51
B	32
A	15

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	15	0	15	0	15	0
B	17	15	32	15	32	0
C	21	15	36	30	51	15
D	19	32	51	32	51	0
E	22	32	54	47	69	15
F	18	51	69	51	69	0
G	15	69	84	69	84	0

The activities with total float = 0 are A, B, D, F and G. They are the critical activities.
Project completion time = 84 weeks.

Problem 3:

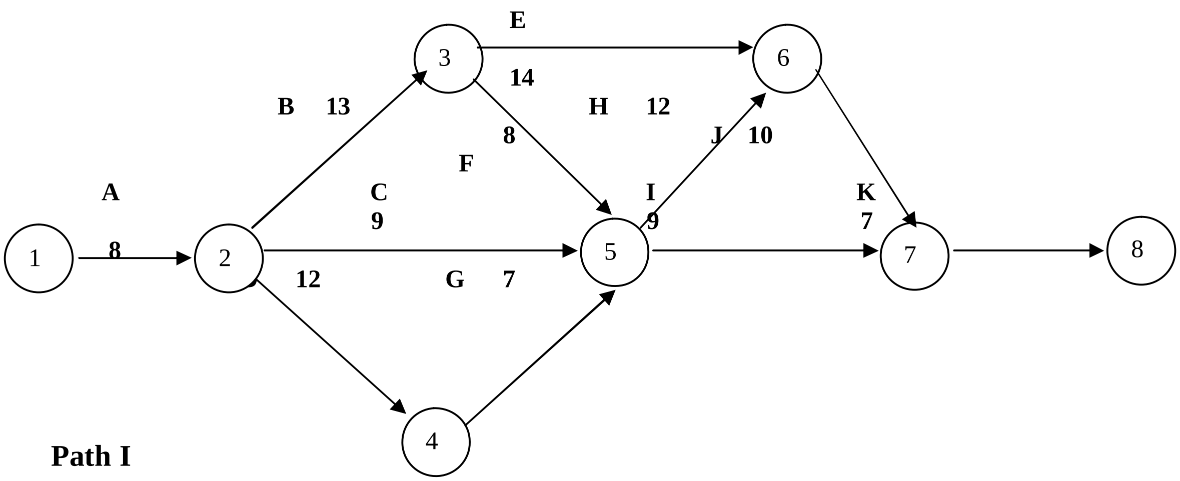
Consider a project with the following details:

Name of Activity	Predecessor Activity	Duration (Weeks)
A	-	8
B	A	13
C	A	9
D	A	12
E	B	14
F	B	8
G	D	7
H	C, F, G	12
I	C, F, G	9
J	E, H	10
K	I, J	7

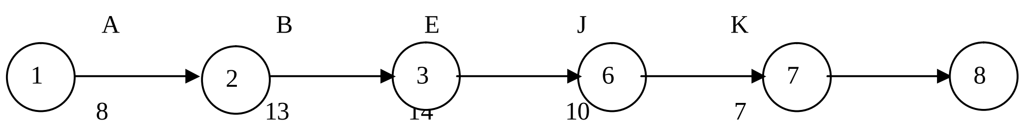
Determine the earliest and latest times, the total float for each activity, the critical activities, the slacks of the events and the project completion time.

Solution:

The following network diagram is got for the given project:

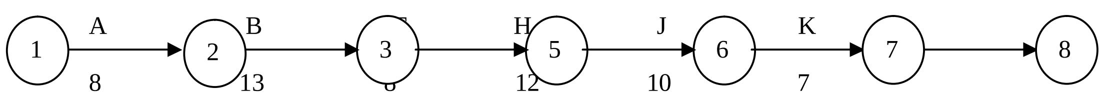


Path I



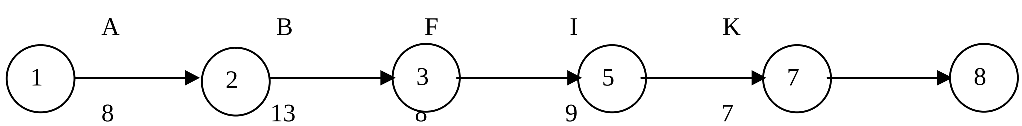
Time of the path = 8 + 13 + 14 + 10 + 7 = 52 weeks.

Path II



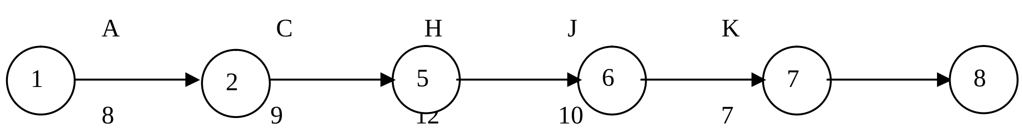
Time of the path = 8 + 13 + 8 + 12 + 10 + 7 = 58 weeks.

Path III



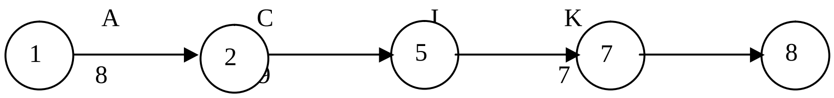
Time of the path = 8 + 13 + 8 + 9 + 7 = 45 weeks.

Path IV



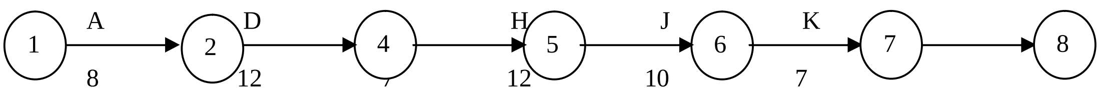
Time of the path = 8 + 9 + 12 + 10 + 7 = 46 weeks.

Path V



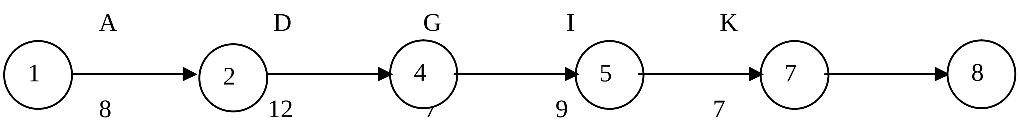
Time of the path = 8 + 9 + 9 + 7 = 33 weeks.

Path VI



Time of the path = 8 + 12 + 7 + 12 + 10 + 7 = 56 weeks.

Path VII



Time of the path = 8 + 12 + 7 + 9 + 7 = 43 weeks.

Compare the times for the three paths. Maximum of {52, 58, 45, 46, 33, 56, 43} = 58.
We see that the maximum time of a path is 58 weeks.

Forward pass:

Calculation of Earliest Time of Occurrence of Events

Node	Earliest Time of Occurrence of Node
1	0
2	Time for Node 1 + Time for Activity A = 0 + 8 = 8
3	Time for Node 2 + Time for Activity B = 8 + 13 = 21
4	Time for Node 2 + Time for Activity D = 8+ 12 = 20
5	Max {Time for Node 2 + Time for Activity C, Time for Node 3 + Time for Activity F, Time for Node 4 + Time for Activity G} = Max { 8 + 9, 21 + 8 , 20 + 7 } = Max {17, 29, 27} = 29
6	Max {Time for Node 3 + Time for Activity E, Time for Node 5 + Time for Activity H} = Max {21 + 14 , 29 + 12} = Max {35, 41} = 41
7	Max {Time for Node 5 + Time for Activity I, Time for Node 6 + Time for Activity J} = Max {29 + 9, 41 + 10} = Max {38, 51} = 51
8	Time for Node 7 + Time for Activity J = 51+ 7 = 58

Earliest Start Times of the activities

Activity	Earliest Start Time (Weeks)
A	0
B	8
C	8
D	8
E	21
F	21
G	20
H	29
I	29
J	41
K	51

Backward pass:

Calculation of Latest Allowable Time of Occurrence of Events

Node	Latest Allowable Time of Occurrence of Node
8	Maximum time of a path in the network = 58
7	Time for Node 8 - Time for Activity K = 58 -7 = 51
6	Time for Node 7 - Time for Activity J = 51 -10 = 41

5	Min {Time for Node 6 - Time for Activity H, Time for Node 7 - Time for Activity I} = Min {41 - 12, 51 - 9} = Min {29, 42} = 29
4	Time for Node 5 - Time for Activity G = 29 - 7 = 22
3	Min {Time for Node 5 - Time for Activity F, Time for Node 6 - Time for Activity E} = Min {29 - 8, 41 - 14} = Min {21, 27} = 21
2	Min {Time for Node 3 - Time for Activity B, Time for Node 4 - Time for Activity D, Time for Node 5 - Time for Activity C} = Min {21 - 13, 22 - 12, 29 - 9} = Min {8, 10, 20} = 8
1	Time for Node 2 - Time for Activity A = 8 - 8 = 0

Latest Finish Times of the activities

Activity	Latest Finish Time (Weeks)
K	58
J	51
I	51
H	41
G	29
F	29
E	41
D	22
C	29
B	21
A	8

Calculation of Total Float for each activity:

Activity	Duration (Weeks)	Earliest Start Time	Earliest Finish Time	Latest Start Time	Latest Finish Time	Total Float = Latest Finish Time - Earliest Finish Time
A	8	0	8	0	8	0
B	13	8	21	8	21	0
C	9	8	17	20	29	12
D	12	8	20	10	22	2
E	14	21	35	27	41	6
F	8	21	29	21	29	0

G	7	20	27	22	29	2
H	12	29	41	29	41	0
I	9	29	38	42	51	13
J	10	41	51	41	51	0
K	7	51	58	51	58	0

The activities with total float = 0 are A, B, F, H, J and K. They are the critical activities.
Project completion time = 58 weeks.

Calculation of slacks of the events

Slack of an event = Latest Allowable Time of Occurrence of the event - Earliest Expected Time of Occurrence of that event.

Event (Node)	Earliest Expected Time of Occurrence of Event	Latest Allowable Time of Occurrence of Event	Slack of the Event
1	0	0	0
2	8	8	0
3	21	21	0
4	20	22	2
5	29	29	0
6	41	41	0
7	51	51	0
8	58	58	0

Interpretation:

On the basis of the slacks of the events, it is concluded that the occurrence of event 4 may be delayed upto a maximum period of 2 weeks while no other event cannot be delayed.

QUESTIONS

1. Explain the terms: The earliest and latest times of the activities of a project.
2. Explain the procedure to find the earliest expected time of an event.
3. Explain the procedure to find the latest allowable time of an event.
4. What is meant by the slack of an activity? How will you determine it?
5. Consider the project with the following details:

activity	Duration (weeks)
1→2	1
2→3	3

2→4	7
3→4	5
3→5	8
4→5	4
5→6	1

Determine the earliest and the latest times of the activities. Calculate the total float for each activity and the slacks of the events.

CRASHING OF A PROJECT

LESSON OUTLINE

- The idea of crashing of a project
- The criterion of selection of an activity for crashing
- Numerical problems

LEARNING OBJECTIVES

After reading this lesson you should be able to

- understand the concept of crashing of a project
- choose an activity for crashing
- work out numerical problems

THE MEANING OF CRASHING:

The process of shortening the time to complete a project is called crashing and is usually achieved by putting into service additional labour or machines to one activity or more activities. Crashing involves more costs. A project manager would like to speed up a project by spending as minimum extra cost as possible. Project crashing seeks to minimize the extra cost for completion of a project before the stipulated time.

STEPS IN PROJECT CRASHING:

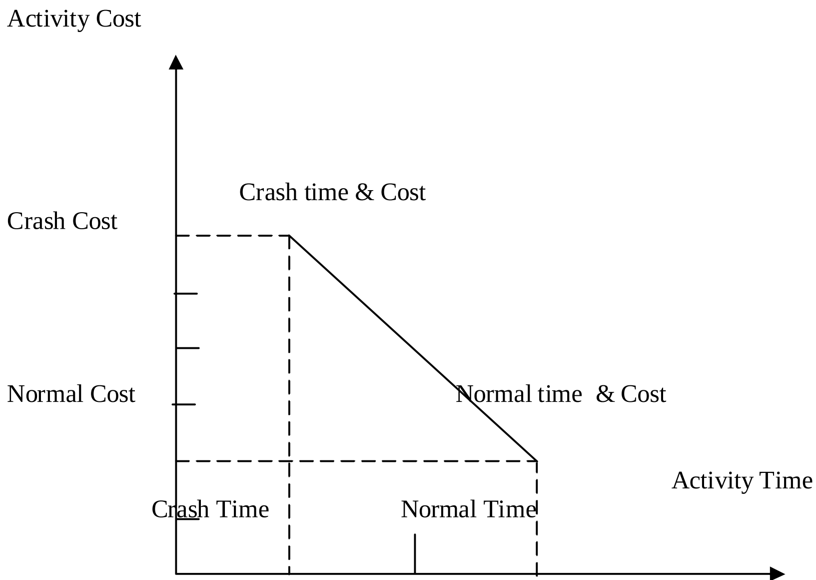
Assumption: It is assumed that there is a linear relationship between time and cost.

Let us consider project crashing by the critical path method. The following four-step procedure is adopted.

Step 1: Find the critical path with the normal times and normal costs for the activities and identify the critical activities.

Step 2: Find out the crash cost per unit time for each activity in the network. This is calculated by means of the following formula.

$$Crash\ \cos t \big/ Time\ \ period = \frac{Crash\ \cos t - Normal\ \cos t}{Normal\ time - Crash\ \ time}$$



Step 3: Select an activity for crashing. The **criteria for the selection** is as follows:
Select the activity on the critical path with the smallest crash cost per unit time. Crash this activity to the maximum units of time as may be permissible by the given data.

Crashing an activity requires extra amount to be spent. However, even if the company is prepared to spend extra money, the activity time cannot be reduced beyond a certain limit in view of several other factors.

In step 1, we have to note that reducing the time of on activity along the critical path alone will reduce the completion time of a project. Because of this reason, we select an activity along the critical path for crashing.

In step 3, we have to consider the following question:

If we want to reduce the project completion time by one unit, which critical activity will involve the least additional cost?

On the basis of the least additional cost, a critical activity is chosen for crashing. If there is a tie between two critical activities, the tie can be resolved arbitrarily.

Step 4: After crashing an activity, find out which is the critical path with the changed conditions. Sometimes, a reduction in the time of an activity in the critical path may cause a non-critical path to become critical. If the critical path with which we started is still the longest path, then go to Step 3. Otherwise, determine the new critical path and then go to Step 3.

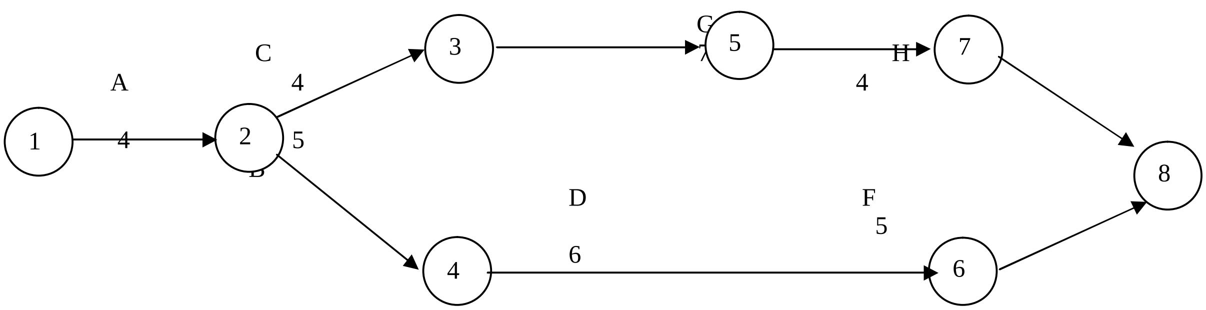
Problem 1: A project has activities with the following normal and crash times and cost:

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	4	3	8,000	9,000
B	A	5	3	16,000	20,000
C	A	4	3	12,000	13,000
D	B	6	5	34,000	35,000
E	C	6	4	42,000	44,000
F	D	5	4	16,000	16,500
G	E	7	4	66,000	72,000
H	G	4	3	2,000	5,000

Determine a crashing scheme for the above project so that the total project time is reduced by 3 weeks.

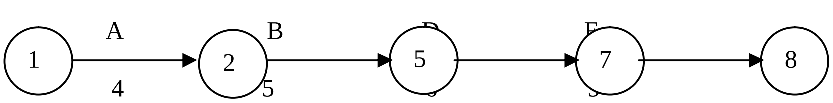
Solution:

We have the following network diagram for the given project with **normal costs**:



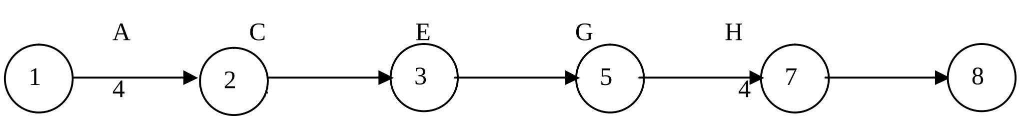
Beginning from the Start Node and terminating with the End Node, there are two paths for the network as detailed below:

Path I:



The time for the path = 4 + 5 + 6 + 5 = 20 weeks.

Path II:



The time for the path = 4 + 4 + 6 + 7 + 4 = 25 weeks.

Maximum of {20, 25} = 25.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. The non-critical activities are B, D and F.

Given that the normal time of activity A is 4 weeks while its crash time is 3 weeks. Hence the time of this activity can be reduced by one week if the management is prepared to spend an additional amount. However, the time cannot be reduced by more than one week even if the management may be prepared to spend more money. The normal cost of this activity is Rs. 8,000 whereas the crash cost is Rs. 9,000. From this, we see that crashing of activity A by one week will cost the management an extra amount of Rs. 1,000. In a similar fashion, we can work out the crash cost per unit time for the other activities also. The results are provided in the following table.

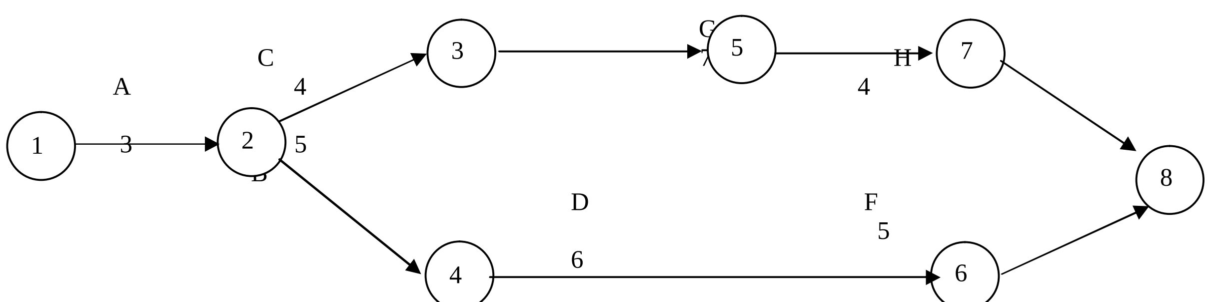
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	4	3	8,000	9,000	1,000	1	1,000
B	5	3	16,000	20,000	4,000	2	2,000
C	4	3	12,000	13,000	1,000	1	1,000
D	6	5	34,000	35,000	1,000	1	1,000
E	6	4	42,000	44,000	2,000	2	1,000
F	5	4	16,000	16,500	500	1	500
G	7	4	66,000	72,000	6,000	1	6,000
H	4	3	2,000	5,000	3,000	1	3,000

A non-critical activity can be delayed without delaying the execution of the whole project. But, if a critical activity is delayed, it will delay the whole project. Because of this reason, we have to select a critical activity for crashing. Here we have to choose one of the activities A, C, E, G and H The crash cost per unit time works out as follows:

Rs. 1,000 for A; Rs. 1,000 for C; Rs. 1,000 for E; Rs. 6,000 for G; Rs. 3,000 for H.

The maximum among them is Rs. 1,000. So we have to choose an activity with Rs.1,000 as the crash cost per unit time. However, there is a tie among A, C and E. The tie can be resolved arbitrarily. Let us select A for crashing. We reduce the time of A by one week by spending an extra amount of Rs. 1,000.

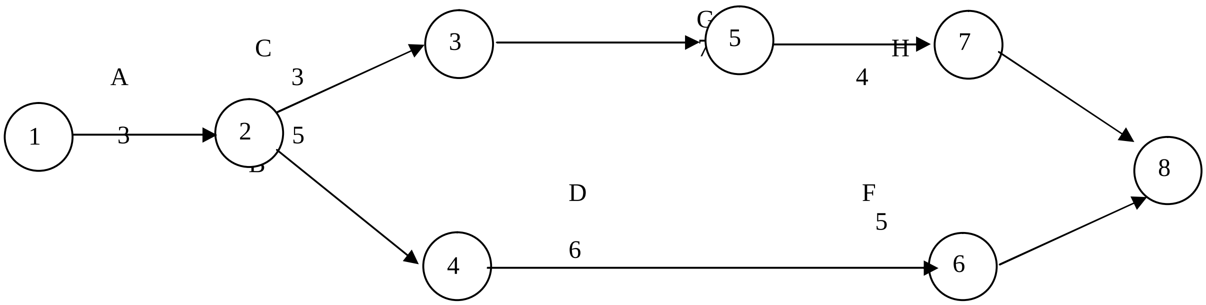
After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 3 + 5 + 6 + 5 = 19 weeks.
The time for Path II = 3 + 4 + 6 + 7 + 4 = 24 weeks.
Maximum of {19, 24} = 24.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. However, the time for A cannot be reduced further. Therefore, we have to consider C, E, G and H for crashing. Among them, C and E have the least crash cost per unit time. The tie between C and E can be resolved arbitrarily. Suppose we reduce the time of C by one week with an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The time for Path I = 3 + 5 + 6 + 5 = 19 weeks.
The time for Path II = 3 + 3 + 6 + 7 + 4 = 23 weeks.
Maximum of {19, 23} = 23.

Therefore Path II is the critical path and the critical activities are A, C, E, G and H. Now the time for A or C cannot be reduced further. Therefore, we have to consider E, G and H for crashing. Among them, E has the least crash cost per unit time. Hence we reduce the time of E by one week with an extra cost of Rs. 1,000.

By the given condition, we have to reduce the project time by 3 weeks. Since this has been accomplished, we stop with this step.

Result: We have arrived at the following crashing scheme for the given project:

Reduce the time of A, C and E by one week each.
Project time after crashing is 22 weeks.
Extra amount required = 1,000 + 1,000 + 1,000 = Rs. 3,000.

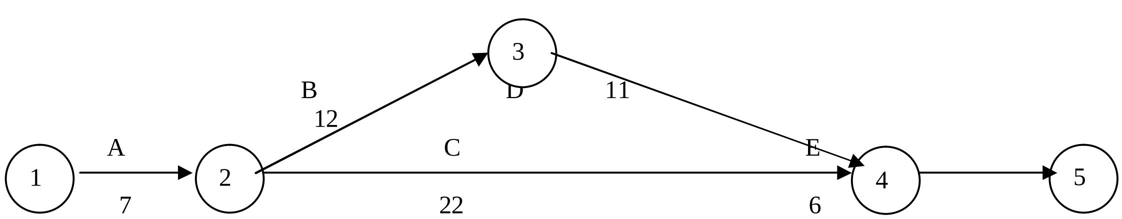
Problem 2:

The management of a company is interested in crashing of the following project by spending an additional amount not exceeding Rs. 2,000. Suggest how this can be accomplished.

Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	7	6	15,000	18,000
B	A	12	9	11,000	14,000
C	A	22	21	18,500	19,000
D	B	11	10	8,000	9,000
E	C, D	6	5	4,000	4,500

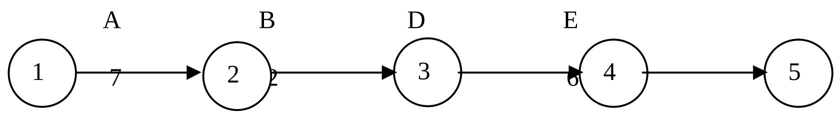
Solution:

We have the following network diagram for the given project with **normal costs**:



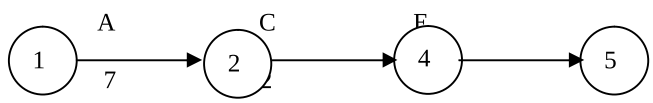
There are two paths for this project as detailed below:

Path I:



The time for the path = 7 + 12 + 11 + 6 = 36 weeks.

Path II:



The time for the path = 7 + 22 + 6 = 35 weeks.

Maximum of {36, 35} = 36.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C.

The crash cost per unit time for the activities in the project are provided in the following table.

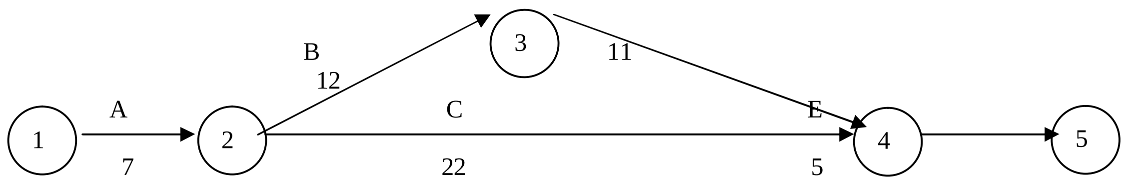
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	7	6	15,000	18,000	3,000	1	3,000
B	12	9	11,000	14,000	3,000	3	1,000
C	22	21	18,500	19,000	500	1	500
D	11	10	8,000	9,000	1,000	1	1,000
E	6	5	4,000	4,500	500	1	500

We have to choose one of the activities A, B, D and E for crashing. The crash cost per unit time is as follows:

Rs. 3,000 for A; Rs. 1,000 for B; Rs. 1,000 for D; Rs. 500 for E.

The least among them is Rs. 500. So we have to choose the activity E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



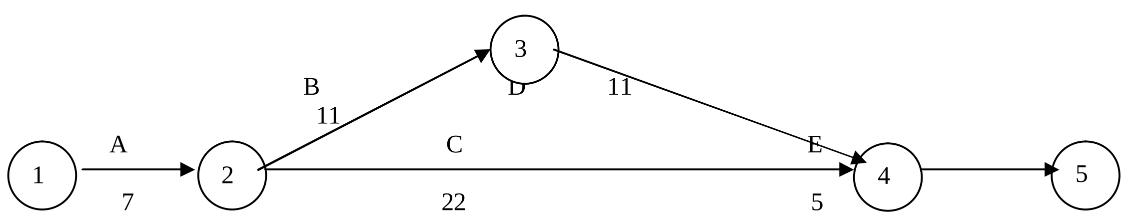
The revised time for Path I = 7 + 12 + 11 + 5 = 35 weeks.

The time for Path II = 7 + 22 + 5 = 34 weeks.

Maximum of {35, 34} = 35.

Therefore Path I is the critical path and the critical activities are A, B, D and E. The non-critical activity is C. The time of E cannot be reduced further. So we cannot select it for crashing. Next B and have the smallest crash cost per unit time. Let us select B for crashing. Let us reduce the time of E by one week at an extra cost of Rs. 1,000.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 7 + 11 + 11 + 5 = 34 weeks.

The time for Path II = 7 + 22 + 5 = 34 weeks.

Maximum of {34, 34} = 34.

Since both paths have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time. In path I, the activities are A, B, D and E. In path II, the activities are A, C and E.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this state has been met, we stop with this step.

Result: The following crashing scheme is suggested for the given project:
Reduce the time of E, B and C by one week each.

Project time after crashing is 33 weeks.

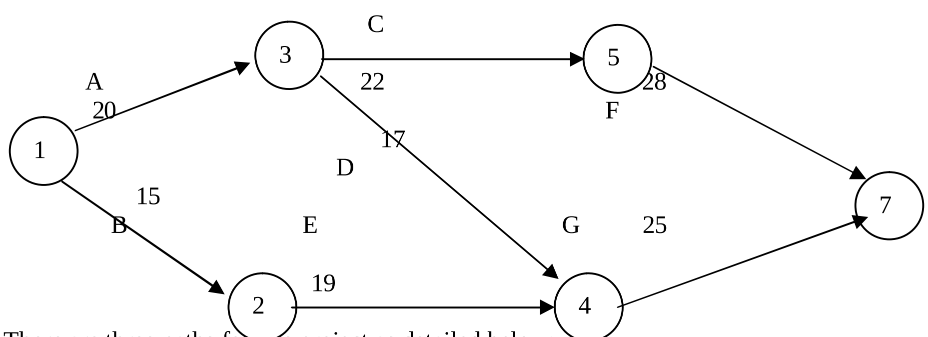
Extra amount required = 500 + 1,000 + 500 = Rs. 2,000.

Problem 3:

The manager of a company wants to apply crashing for the following project by spending an additional amount not exceeding Rs. 2,000. Offer your suggestion to the manager.

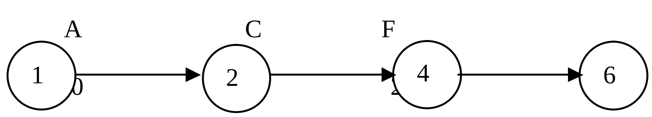
Activity	Predecessor Activity	Normal Time (Weeks)	Crash Time (Weeks)	Normal Cost (Rs.)	Crash Cost (Rs.)
A	-	20	19	8,000	10,000
B	-	15	14	16,000	19,000
C	A	22	20	13,000	14,000
D	A	17	15	7,500	9,000
E	B	19	18	4,000	5,000
F	C	28	27	3,000	4,000
G	D, E	25	24	12,000	13,000

Solution:
We have the following network diagram for the given project with **normal costs**:



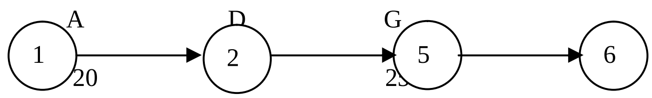
There are three paths for this project as detailed below:

Path I:



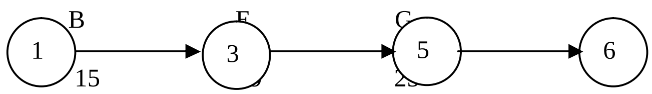
The time for the path = 20 + 22 + 28= 70 weeks.

Path II:



The time for the path = 20 + 17 + 25= 62 weeks.

Path III:



The time for the path = 15+19 +25 = 69 weeks.
Maximum of {70, 62, 69} = 70.
Therefore Path I is the critical path and the critical activities are A, C and F. The non-critical activities are B, D, E and G.
The crash cost per unit time for the activities in the project are provided in the following table

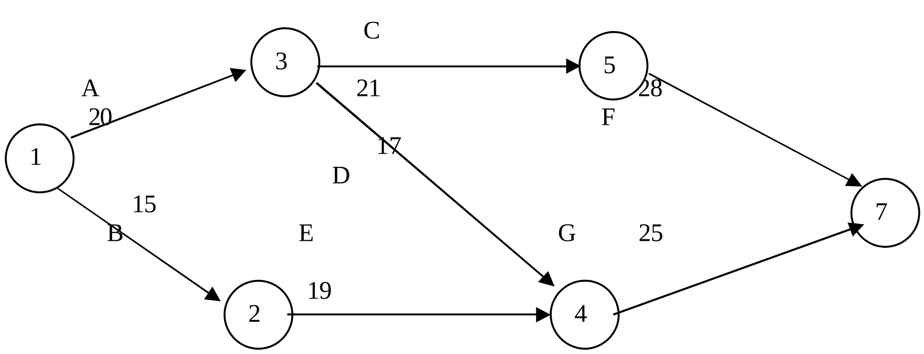
Activity	Normal Time	Crash Time	Normal Cost	Crash Cost	Crash cost - Normal Cost	Normal Time - Crash Time	Crash Cost per unit time
A	20	19	8,000	10,000	2,000	1	2,000
B	15	14	16,000	19,000	3,000	1	3,000
C	22	20	13,000	14,000	1,000	2	500
D	17	15	7,500	9,000	1,500	2	750
E	19	18	4,000	5,000	1,000	1	1,000
F	28	27	3,000	4,000	1,000	1	1,000
G	25	24	12,000	13,000	1,000	1	1,000

We have to choose one of the activities A, C and F for crashing. The crash cost per unit time is as follows:

Rs. 2,000 for A; Rs. 500 for C; Rs. 1,000 for F.

The least among them is Rs. 500. So we have to choose the activity C for crashing. We reduce the time of C by one week by spending an extra amount of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 20 + 21 + 28= 69 weeks.

The time for Path II = 20 + 17 + 25= 62 weeks.

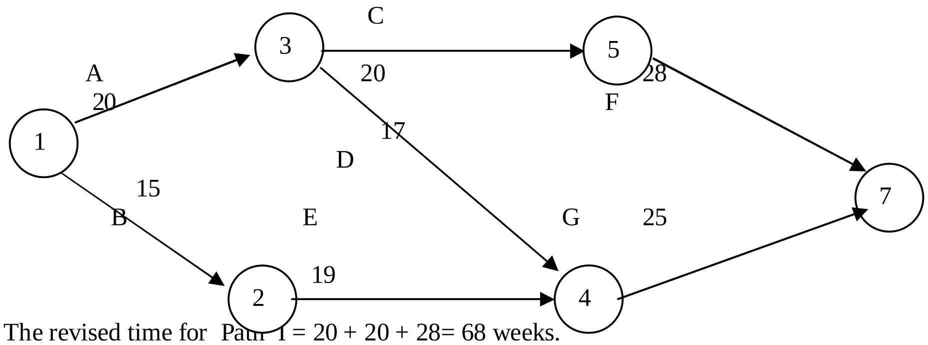
The time for Path III = 15+19 +25 = 69 weeks.
Maximum of {69, 62, 69} = 69.

Since paths I and III have equal times, both are critical paths. So, we can choose an activity for crashing from either of them depending on the least crash cost per unit time.

In path I, the activities are A, C and F. In path III, the activities are B, E and G.

The crash cost per unit time is the least for activity C. So we select C for crashing. Reduce the time of C by one week at an extra cost of Rs. 500.

After this step, we have the following network with the revised times for the activities:



The revised time for Path I = 20 + 20 + 28= 68 weeks.
The time for Path II = 20 + 17 + 25= 62 weeks.

The time for Path III = 15+19 +25 = 69 weeks.
Maximum of {68, 62, 69} = 69.

Therefore path III is the critical activities. Hence we have to select an activity from Path III for crashing. We see that the crash cost per unit time is as follows:

Rs. 3,000 for B; Rs. 1,000 for E; Rs. 1,000 for G.

The least among them is Rs. 1,000. So we can select either E or G for crashing. Let us select E for crashing. We reduce the time of E by one week by spending an extra amount of Rs. 1,000.

By the given condition, the extra amount cannot exceed Rs. 2,000. Since this condition has been reached, we stop with this step.

Result: The following crashing scheme is suggested for the given project:

Reduce the time of C by 2 weeks and that of E by one week.

Project time after crashing is 67 weeks.

Extra amount required = 2 x 500 + 1,000 = Rs. 2,000.

QUESTIONS

1. Explain the concept of crashing of a project.
2. Explain the criterion for selection of an activity for crashing.

Queueing Theory

1 Introduction:

A flow of customers from finite or infinite population towards the service facility forms a **queue (waiting line)** an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, **waiting time** is required either for the service facilities or for the customers arrival. In general, the **queueing system** consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience " Customer waiting" and /or "Server idle time"

2 Queueing System:

A queueing system can be completely described by

(1) the input (arrival pattern)
(2) the service mechanism (service pattern)
(3) The queue discipline and
(4) Customer's behaviour

3 The input (arrival pattern)

The input described the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random manner which is not possible for prediction. Thus the arrival pattern can be described in terms of probabilities and consequently the probability distribution for **inter-arrival** times (the time between two successive arrivals) must be defined. We deal with those Queueing system in which the customers arrive in poisson process. The mean arrival rate is denoted by λ.

4 The Service Mechanism:

This means the arrangement of service facility to serve customers. If there is infinite number of servers, then all the customers are served instantaneously or arrival and there will be no queue. If the number of servers is finite then the customers are served according to a specific order with service time a constant or a random variable. Distribution of service time follows '**Exponential distribution'** defined by

$$f(t) = \lambda e^{-\lambda t}, \quad t > 0$$

The mean Service rate is $E(t) = 1/\lambda$

5 Queueing Discipline:

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

1. First come first served – (FCFS)
2. First in first out – (FIFO)
3. Last in first out – (LIFO)
4. Selection for service in random order (SIRO)

6 Customer's behaviour

1. Generally, it is assumed that the customers arrive into the system one by one. But in some cases, customers may arrive in groups. Such arrival is called **Bulk arrival.**

2. If there is more than one queue, the customers from one queue may be tempted to join another queue because of its smaller size. This behaviour of customers is known as **jockeying.**

3. If the queue length appears very large to a customer, he/she may not join the queue. This property is known as **Balking** of customers.

4. Sometimes, a customer who is already in a queue will leave the queue in anticipation of longer waiting line. This kind of departare is known as **reneging.**

7 List of Variables

The list of variable used in queueing models is give below:

n - No of customers in the system
C - No of servers in the system
$P_n(t)$ – Probability of having n customers in the system at time t.
P_n - Steady state probability of having customers in the system
P_0 - Probability of having zero customer in the system
L_q - Average number of customers waiting in the queue.
L_s - Average number of customers waiting in the system (in the queue and in the service counters)
W_q - Average waiting time of customers in the queue.
W_s - Average waiting time of customers in the system (in the queue and in the service counters)
δ - Arrival rate of customers
μ - Service rate of server
ϕ - Utilization factor of the server
δ eff - Effective rate of arrival of customers
M - Poisson distribution
N - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.
GD - General discipline for service. This may be first in first – serve (FIFS), last-in-first serve (LIFS) random order (Ro) etc.

8 Traffic intensity (or utilization factor)

An important measure of a simple queue is its traffic intensity given by

$$\text{Traffic intensity } \phi = \frac{\text{Mean arrival time}}{\text{Mean service time}} = \frac{\delta}{\mu} \quad (< 1)$$

and the unit of traffic intensity is Erlang

9 Classification of Queueing models

Generally, queueing models can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are

P- Arrival rate distribution ie probability law for the arrival /inter – arrival time.

Q - Service rate distribution, ie probability law according to which the customers are being served.

R - Number of Servers (ie number of service stations)
X - Service discipline
Y - Maximum number of customers permitted in the system.
Z - Size of the calling source of the customers.

A queuing model with the above parameters is written as
(P/Q/R : X/Y/Z)

10 Model 1 : (M/M/1) : (GD/ ∞ / ∞) Model

In this model

(i) the arrival rate follows poisson (M) distribution.
(ii) Service rate follows poisson distribution (M)
(iii) Number of servers is 1
(iv) Service discipline is general disciple (ie GD)
(v) Maximum number of customers permitted in the system is infinite (∞)
(vi) Size of the calling source is infinite (∞)

The steady state equations to obtain, P_n the probability of having customers in the system and the values for Ls, Lq, Ws and Wq are given below.

$$n = 0,1,2,\text{----} \infty \text{ where } \phi = \frac{\delta}{\mu} < 1$$

Ls – Average number of customers waiting in the system
(ie waiting in the queue and in the service station)

$$\boxed{P_n = \phi^n (1-\phi)}$$

$$\boxed{Ls = \frac{\phi}{1-\phi}}$$

$$Lq = Ls - \frac{\delta}{\mu}$$

$$= \frac{\phi}{1-\phi} - \phi$$

$$= \frac{\phi - (1-\phi)\phi}{1-\phi}$$

$$\boxed{Lq = \frac{\phi^2}{1-\phi}}$$

Average waiting time of customers in the system (in the queue and in the service station) $= W_s$

$$= Ws = \frac{L_s}{\delta} = \frac{\phi}{(1-\phi)\delta}$$

$$= \frac{\phi}{1-\phi} \times \frac{1}{\mu\phi} = \frac{1}{\mu(1-\phi)}$$

(Since $\delta = \mu\phi$)

$$= \frac{1}{\mu - \mu\phi}$$

$$= \frac{1}{\mu - \delta}$$

$$\boxed{W_s = \frac{1}{\mu - \delta}}$$

W_q = Average waiting time of customers in the queue.

$$= L_q / \delta = [1/\ \delta]\ [\phi^2 / [1-\phi]]$$

$$= 1/\ \mu\phi\ [\phi^2 / [1-\phi]]$$

$$= \frac{\phi}{\mu - \mu\phi} \quad \text{Since } \mu\phi = \delta$$

$$\boxed{Wq = \frac{\phi}{\mu - \delta}}$$

Example 1:

The arrival rate of customers at a banking counter follows a poisson distibution with a mean of 30 per hours. The service rate of the counter clerk also follows poisson distribution with mean of 45 per hour.

a) What is the probability of having zero customer in the system ?
b) What is the probability of having 8 customer in the system ?
c) What is the probability of having 12 customer in the system ?
d) Find Ls, Lq, Ws and Wq

Solution

Given arrival rate follows poisson distribution with mean =30

$\therefore \delta$= 30 per hour

Given service rate follows poisson distribution with mean = 45

$\therefore \mu$ = 45 Per hour

$\therefore$ Utilization factor $\phi = \delta/\mu$

$= 30/45$

$= 2/3$

$= 0.67$

a) The probability of having zero customer in the system

$P_0 = \phi^0 (1-\phi)$

$= 1-\phi$

$= 1-0.67$

$= 0.33$

b) The probability of having 8 customers in the system

$$P_8 = \phi^8 (1-\phi)$$
$$= (0.67)^8 (1-0.67)$$
$$= 0.0406 \times 0.33$$
$$= 0.0134$$

Probability of having 12 customers in the system is

$$P_{12} = \phi^{12} (1-\phi)$$
$$= (0.67)^{12} (1-0.67)$$
$$= 0.0082 \times 0.33 \quad \mathbf{= 0.002706}$$

$$L_s = \frac{\phi}{1-\phi} = \frac{0.67}{1-0.67}$$

$$= \frac{0.67}{0.33} = 2.03$$

$$\mathbf{= 2\ customers}$$

$$L_q = \frac{\phi^2}{1-\phi} = \frac{(0.67)^2}{1-0.67} = \frac{0.4489}{0.33}$$
$$= 1.36$$

$$= \mathbf{1\ Customer}$$

$$W_s = \frac{1}{\mu - \delta} = \frac{1}{45-30} = \frac{1}{15}$$
$$= \mathbf{0.0666\ hour}$$

$$W_q = \frac{\phi}{\mu - \delta} = \frac{0.67}{45-30} = \frac{0.67}{15}$$

$$= \mathbf{0.4467\ hour}$$

Example 2 :

At one-man barbar shop, customers arrive according to poisson dist with mean arrival rate of 5 per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

(i) Average number of customers in the shop and the average numbers waiting for a haircut.
(ii) The percentage of time arrival can walk in straight without having to wait.
(iii) The percentage of customers who have to wait before getting into the barber's chair.

Solution:-

Given mean arrival of customer $\delta = 5/60 = 1/12$

and mean time for server $\mu = 1/10$

$\therefore \phi = \delta / \mu = [1/12] \times 10 = 10/12$

$= \mathbf{0.833}$

(i) Average number of customers in the system (numbers in the queue and in the service station)

$$L_s = \phi / 1-\phi = 0.83 / 1- 0.83$$
$$= 0.83 / 0.17$$
$$= 4.88$$

= 5 Customers

(ii) The percentage of time arrival can walk straight into barber's chair without waiting is

Service utilization $= \phi\ \%$

$= \delta / \mu\%$

$= 0.833 \times 100$

$\mathbf{=83.3}$

(iii) The percentage of customers who have to wait before getting into the barber's chair = $(1-\phi)\%$

$(1-0.833)\%$ = 0.167×100

= **16.7%**

Example 3 :

Vehicles are passing through a toll gate at the rate of 70 per hour. The average time to pass through the gate is 45 seconds. The arrival rate and service rate follow poisson distibution. There is a complaint that the vehicles wait for a long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 35 seconds if the idle time of the toll gate is less than 9% and the average queue length at the gate is more than 8 vehicle, check whether the installation of the second gate is justified?

Solutions:-

Arrival rate of vehicles at the toll gate δ = 70 per hour

Time taken to pass through the gate = 45 Seconds

Service rate μ = $\dfrac{1 \text{ hours}}{45 \text{ seconds}}$

= 3600/45 = 80

= **80 Vehicles per hour**

$\therefore$ Utilization factor $\phi = \delta/\mu$

= 70 / 80

= **0.875**

(a) Waiting no. of vehicles in the queue is L_q

$L_q = \phi^2 / 1 - \phi$ = $\dfrac{(0.875)^2}{1-0.875}$

= $\dfrac{0.7656}{0.125}$

= **6.125**

= 6 Vehicles

(b) Revised time taken to pass through the gate =30 seconds

$\therefore$ The new service rate after installation of an additional gate = 1 hour/35 Seconds = 3600/35

= **102.68 Vehicles / hour**

$\therefore$ Utilization factor $\phi = \delta / \mu = 70 / 102.86$

= 0.681

Percentage of idle time of the gate = $(1-\phi)\%$

= $(1-0.681)\%$

= 0.319%

= 31.9

= 32%

This idle time is not less than 9% which is expected.

Therefore the installation of the second gate is not justified since the average waiting number of vehicles in the queue is more than 8 but the idle time is not less than 32%. Hence idle time is far greater than the number. of vehicles waiting in the queue.

11 Second Model (M/M/C) : (GD/ ∞/ ∞)Model

The parameters of this model are as follows:

(i) Arrival rate follows poisson distribution

(ii) Service rate follows poisson distribution

(iii) No of servers is C'.

(iv) Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is infinite

Then the steady state equation to obtain the probability of having n customers in the system is

$$P_n = \frac{\phi^n P_o}{n!} \quad , \quad o \le n \le c$$

$$= \frac{\phi^n P_o}{c^{n-c} c!} \quad \text{for } n > c \text{ Where } \phi / c < 1$$

Where $[\delta / \mu c] < 1$ as $\phi = \delta / \mu$

$$\therefore P_0 = \{[\sum_{n=0}^{c-1} \phi^n/n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}$$

where c! = 1 x 2 x 3 x …………….. upto c

$L_q = [\phi^{c+1} / [c\text{-}1! (c - \phi)]] \times P_0$

$= (c\phi P_c) / (c - \phi)^2$

$L_s = L_q + \phi$ and $W_s = W_q + 1 / \mu$

$W_q = L_q / \delta$

Under special conditions $P_o = 1 - \phi$ and $L_q = \phi^{c+1} / c^2$ Where $\phi < 1$ and
$P_o = (c - \phi)(c - 1)! / c^c$
and $L_q = \phi / (c - \phi)$, where $\phi / c < 1$

Example 1:

At a central warehouse, vehicles are at the rate of 24 per hour and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find

(i) P_o and P_3
(ii) L_q, L_s, W_q and W_s

Solution:

Arrival rate δ = 24 per hour
Unloading rate μ = 18 Per hour
No. of unloading crews c=4

$\phi = \delta / \mu$ = 24 / 18**=1.33**

(i) $P_0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}$

$= \{[\sum_{n=0}^{3} (1.33)^n/n!] + (1.33)^4/(4! [1 - (1.33)/ 4])\}^{-1}$

$= \{ (1.33)^0 / = 0! + (1.33)^1 / 1! + (1.33)^2 / 2! + (1.33)^3 / 3! +$
$(1.33)^4 / 24! [1 - (1.33)/ 4] \}^{-1}$

$= [1 + 1.33 + 0.88 + 0.39 + 3.129/16.62]^{-1}$

$= [3.60 + 0.19]^{-1} = [3.79]^{-1}$

= 0.264

We know $P_n = (\phi^n / n!) P_o$ for $0 \le n \le c$

$\therefore P_3 = (\phi^3 / 3!) P_o$ Since $0 \le 3 \le 4$

$= [(1.33)^3 / 6] \times 0.264$

= 2.353 x 0.044

= 0.1035

(ii) $L_q = \dfrac{\phi^{c+1} \text{ X } P_0}{(c - 1)! (c - \phi)^2}$

$= \dfrac{(1.33)^5 \text{ X } 0.264}{3! \text{ X } (4 - 1.33)^2}$

$= \dfrac{(4.1616) \text{ X } 0.264}{6 \text{ X } (2.77)^2}$

$= \dfrac{(4.1616) \times 0.264}{46.0374}$

$= 1.099 / 46.0374$

$= 0.0239$

= 0.0239 Vehicles

$L_s = L_q + \phi = 0.0239 + 1.33$

= 1.3539 Vehicles

$W_q = L_q / \delta = 0.0239 / 24$

= 0.000996 hrs

$W_s = W_q + 1 / \mu = 0.000996 + 1/18$

$= 0.000996 + 0.055555$

$= 0.056551$ hours.

Example 2 :-

A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if the people arrive in poisson fashion at the rate of 10 per hour

a) What is the probability of having to wait for service?
b) What is the expected percentage of idle time for each girl?
c) If a customer has to wait, what is the expected length of his waiting time?

Solution:-

$$P_0 = \{[\sum_{n=0}^{c-1} \phi^n/n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}$$

Where $\phi = \delta / \mu$ $\therefore$ given arrival rate = 10 per hour

$\delta = 10 / 60 = 1 / 6$ per minute

Service rate = 4 minutes

$\therefore \mu = 1 / 4$ person per minute

Hence $\phi = \delta / \mu = (1 / 6) \times 4 = 2 / 3$

= 0.67

$$P_0 = \{[\sum_{n=0}^{1} \phi^n/n!] + (0.67)^2 / (2! [1 - 0.67/2])\}^{-1}$$

$=[1 + (\phi / 1!)] + 0.4489 / (2 – 0.67)]^{-1}$

$=[1 + 0.67 + 0.4489 / (1.33)]^{-1}$

$=[1 + 0.67 + 0.34]^{-1}$

$=[2.01]^{-1}$

= 1 / 2

The Probability of having to wait for the service is

P (w > 0)

$= \dfrac{\phi^c}{c! [1 - \phi /c]} \times P_0$

$= \dfrac{0.67^2 \times (1 / 2)}{2! [1 – 0.67 /2]}$

$= 0.4489 / 2.66$

= 0.168

b) The probability of idle time for each girl is

$= 1 - P (w > 0)$

$= 1\text{-}1/3$
$\mathbf{= 2/3}$

$\therefore$ Percentage of time the service remains idle = 67% approximately

c) The expected length of waiting time (w/w>0)

$= 1 / (c\,\mu - \delta)$
$= 1 / [(1/2) - (1/6)]$
= 3 minutes

Examples 3 :

A petrol station has two pumps. The service time follows the exponential distribution with mean 4 minutes and cars arrive for service in a poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time the pump remains idle?

Solution: **Given c=2 The arrival rate = 10 cars per hour.**

$\therefore \delta = 10 / 60 = 1/6$ car per minute

Service rate = 4 minute per cars.

Ie $\mu = 1/4$ car per minute.

$\phi = \delta / \mu = (1/6) / (1/4)$
$= 2/3$
$\mathbf{= 0.67}$

Proportion of time the pumps remain busy

$= \phi / c = 0.67 / 2$
$= 0.33$
$= 1/3$

$\therefore$ The proportion of time, the pumps remain idle

=1 – proportion of the pumps remain busy
$= 1\text{-}1/3 = 2/3$

$$P_0 = \{[\sum_{n=0}^{c-1} \phi^n/n!] + \phi^c / (c!\,[1 - \phi/c])\}^{-1}$$

$=[(0.67)^0 / 0!) + (0.67)^1 / 1!) + (0.67)^2 / 2!)[1\text{-} (0.67 / 2)^1]^{-1}$
$=[1 + 0.67 + 0.4489 / (1.33)]^{-1}$
$=[1 + 0.67 + 0.33]^{-1}$
$=[2]^{-1}$
$\mathbf{=1/2}$

Probability that a customer has to wait for service

$= p\,[w>0]$

$$= \frac{\phi^c \times P_0}{[c!\,[1\text{-}\phi/c]} = \frac{(0.67)^2 \times 1/2}{[2![1 - 0.67/2]}$$

$$= \frac{0.4489}{1.33\text{x}2} = \frac{0.4489}{2.66}$$

$\mathbf{= 0.1688}$

Simulation :

The representation of reality in some physical form or in some form of Mathematical equations are called Simulations . Simulations are imitation of reality.

For example :

1. Children cycling park with various signals and crossing is a simulation of a read model traffic system
2. Planetarium
3. Testing an air craft model in a wind tunnel.

5.2.2. Need for simulation :

Consider an example of the queueing system, namely the reservation system of a transport corporation. The elements of the system are booking counters (servers) and waiting customers (queue). Generally the arrival rate of customers follow a Poisson distribution and the service time follows exponential distribution. Then the queueing model (M/M/1) : (GD/∞ /∞) can be used to find the standard results. But in reality, the following combinations of distributions my exist.

1. Arrived rate does not follow Poisson distribution, but the service rate follows an exponential distribution.
2. Arrival rate follows a Poisson distribution and the service rate does not follow exponential distribution.
3. Arrival rate does not follows poisson distribution and the service time also does not follow exponential distribution. In each of the above cases, the standard model (M/M/1) : (G/D/∞ /∞) cannot be used. The last resort to find the solution for such a queueing problem is to use simulation.

5.2.3. Some advantage of simulation :

1. Simulation is Mathematically less complicated
2. Simulation is flexible
3. It can be modified to suit the changing environments.
4. It can be used for training purpose
5. It may be less expensive and less time consuming in a quite a few real world situations.

5.2.4. Some Limitations of Simulation :

1. Quantification or Enlarging of the variables maybe difficult.
2. Large number of variables make simulations unwieldy and more difficult.
3. Simulation may not. Yield optimum or accurate results.
4. Simulation are most expensive and time consuming model.
5. We cannot relay too much on the results obtained from simulation models.

5.2.5. Steps in simulation :

1. Identify the measure of effectiveness.
2. Decide the variables which influence the measure of effectiveness and choose those variables, which affects the measure of effectiveness significantly.
3. Determine the probability distribution for each variable in step 2 and construct the cumulative probability distribution.
4. Choose an appropriate set of random numbers.
5. Consider each random number as decimal value of the cumulative probability distribution.
6. Use the simulated values so generated into the formula derived from the measure of effectiveness.
7. Repeat steps 5 and 6 until the sample is large enough to arrive at a satisfactory and reliable decision.

5.2.6. Uses of Simulation

Simulation is used for solving

1.Inventory Problem
2. Queueing Problem
3. Training Programmes etc.

Example :

Customers arrive at a milk booth for the required service. Assume that inter – arrival and service time are constants and given by 1.5 and 4 minutes respectively. Simulate the system by hand computations for 14 minutes.

(i) What is the waiting time per customer?
(ii) What is the percentage idle time for the facility?
(Assume that the system starts at t = 0)

Solution :

First customer arrives at the service center at t = 0
∴ His departure time after getting service = 0 + 4 = 4 minutes.
Second customer arrives at time t = 1.5 minutes
∴ he has to wait = 4 – 1.5 = 2.5 minutes.
Third customer arrives at time t = 3 minutes
∴ he has to wait for = 8-3 = 5 minutes
Fourth customer arrives at time t = 4.5 minutes and he has to wait for 12 – 4.5 = 7.5 minutes.
During this 4.5 minutes, the first customer leaves in 4 minutes after getting service and the second customer is getting service.
Fifth customer arrives at t = 6 minutes
∴ he has to wait 14 – 6 = 8 minutes
Sixth customer arrives at t = 7.5 minutes
∴he has to wait 14 – 7.5 = 6.5 minutes
Seventh customer arrives at t = 9 minutes
∴he has to wait 14 – 9 = 5 minutes
During this 9 minutes the second customer leaves the service in 8^{th} minute and third person is to get service in 9^{th} minute.
Eighth customer arrives at t = 10.5 minutes
∴he has to wait 14 – 10.5 = 3.5 minutes
Nineth customer arrives at t = 12 minutes
∴he has to wait 14 – 12 = 2 minutes
But by 12^{th} minute the third customer leaves the Service
10^{th} Customer arrives at t = 13.5 minutes
∴he has to wait 14-13.5 = 0.5 minute

From this simulation table it is clear that

(i) Average waiting time for 10 customers

$$= \frac{2.5+5+7.5+8+6.5+5.0+3.5+2+0.5}{10}$$

$$= \frac{40.5}{10} = 4.05$$

(ii) Average waiting time for 9 customers who are in waiting for service $\frac{40.5}{9}$ = 4.5 minutes.

But the average service time is 4 minutes which is less that the average waiting time, the percentage of idle time for service = 0%

Exercise :

1. The arrival rate of customers at a banking counter follows a poisson distribution with a mean of 45 per hors. The service rate of the counter clerk also poisson distribution with a mean of 60 per hours.

(a) What is the probability of having Zero customer in the system (P_0).
(b) What is the probability of having 5 customer in the system (P_5).
(c) What is the probability of having 10 customer in the system (P_{10}).
(d) Find Ls, Lq, Ws and Wq

2. Vehicles pass through a toll gate at a rate of 90 per hour. The average time to pass through the gate is 36 seconds. The arrival rate and service rate follow poisson distribution. There is a complaint the vehicles wait for long duration. The authorities are willing to install one more gate to reduce the average time to pass through the toll gate to 30 seconds if the idle time of the toll gate is less than 10% and the average queue length at the gate is more than 5 vehicles. Vehicle whether the installation of second gate is justified?

3. At a central ware house, vehicles arrive at the rate of 24 per hours and the arrival rate follows poisson distribution. The unloading time of the vehicles follows exponentional distribution and the unloading rate is 18 vehicles per hour. There are 4 unloading crews. Find the following.

a) P_0 and P_3
b) L_q, L_s, W_q and W_s

4. Explain Queneing Discipline
5. Describe the Queueing models (M/M/1) : (GD/∞ /∞) and (M/M/C) : (GD/ ∞ / ∞)

6. Cars arrive at a drive-in restaurant with mean arrival rate of 30 cars per hors and the service rate of the cars is 22 per hors. The arrival rate and the service rate follow poisson distribution. The number parking space for cars is only 5. Find the standard results.
(Ans L_q = 2.38 cars, L_s = 3.3133 Cars, W_q = 0.116 hors and W_s = 0.1615 hors)

7. In a harbour, ship arrive with a mean rate of 24 per week. The harbour has 3 docks to handle unloading and loading of ships. The service rate of individual dock is 12 per week. The arrival rate and the service rate follow poisson distribution. At any point of time, the maximum No. of ships permitted in the harbour is 8. Find P_0, Lq, L_s, W_q, W_s
(Ans P_0 = 0.1998, L_q = 1.0371 ships, L_s = 2.9671 ships, W_q = 0.04478 week and W_s = 0.1281 week)

8. Define simulation and its advantages.
9. Discuss the steps of simulation.

5.3. Replacement models

5.3.1. Introduction:

The replacement problems are concerned with the situations that arise when some items such as men, machines and usable things etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

If a firm wants to survive the competition it has to decide on whether to replace the out dated equipment or to retain it, by taking the cost of maintenance and operation into account. There are two basic reasons for considering the replacement of an equipment.

They are

(i) Physical impairment or malfunctioning of various parts.
(ii) Obsolescence of the equipment.

The physical impairment refers only to changes in the physical condition of the equipment itself. This will lead to decline in the value of service rendered by the equipment, increased operating cost of the equipments, increased maintenance cost of the equipment or the combination of these costs. Obsolescence is caused due to improvement in the existing Tools and machinery mainly when the technology becomes advanced therefore, it becomes uneconomical to continue production with the same equipment under any of the above situations. Hence the equipments are to be periodically replaced. Some times, the capacity of existing facilities may be in adequate to meet the current demand. Under such cases, the following two alternatives will be considered.

1. Replacement of the existing equipment with a new one
2. Argument the existing one with an additional equipments.

5.3.2 <u>Type of Maintenance</u>

Maintenance activity can be classified into two types

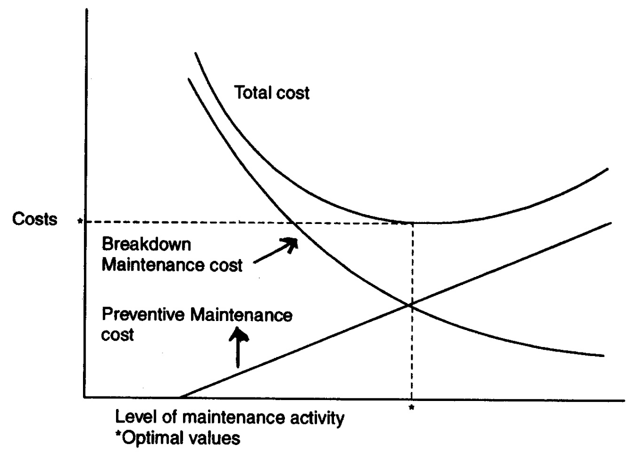
i) Preventive Maintenance
ii) Breakdown Maintenance

Preventive maintenance (PN) is the periodical inspection and service which are aimed to detect potential failures and perform minor adjustments a requires which will prevent major operating problem in future. Breakdown maintenance is the repair which is generally done after the equipment breaks down. It is offer an emergency which will have an associated penalty in terms of increasing the cost of maintenance and downtime cost of equipment, Preventive maintenance will reduce such costs up-to a certain extent . Beyond that the cost of preventive maintenance will be more when compared to the cost of the breakdown maintenance.

Total cost = Preventive maintenance cost + Breakdown maintenance cost.

This total cost will go on decreasing up-to P with an increase in the level of maintenance up-to a point, beyond which the total cost will start increasing from P. The level of maintenance corresponding to the minimum total cost at P is the Optional level of maintenance this concept is illustrated in the follows diagram

N



M

The points M and N denote optimal level of maintenance and optimal cost respectively

5.3.3 <u>Types of replacement problem</u>

The replacement problem can be classified into two categories.

i) Replacement of assets that deteriorate with time (replacement due to gradual failure, due to wear and tear of the components of the machines) This can be further classified into the following types.
   a) Determination of economic type of an asset.
   b) Replacement of an existing asset with a new asset.

ii) simple probabilistic model for assets which will fail completely (replacement due to sudden failure).

5.3.4. <u>Determination of Economic Life of an asset</u>

Any asset will have the following cost components

i) Capital recovery cost (average first cost), Computed form the first cost (Purchase price) of the asset.

ii) Average operating and maintenance cost.

iii) Total cost which is the sum of capital recovery cost (average first cost) and average operating and maintenance cost.

A typical shape of each of the above cost with respect to life of the asset is shown below

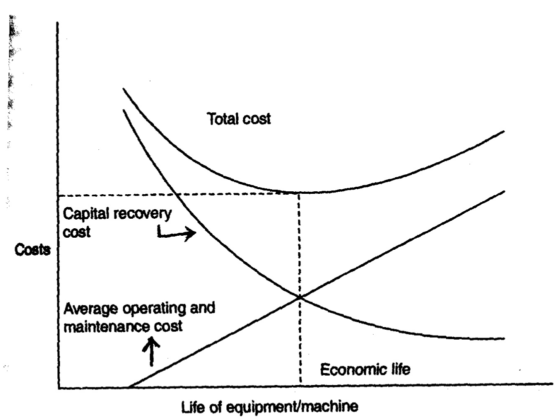


Fig. 13.2 Chart showing economic life.

From figure, when the life of the machine increases, it is clear that the capital recovery cost (average first cost) goes on decreasing and the average operating and maintenance cost goes on increasing. From the beginning the total cost goes on decreasing upto a particular life of the asset and then it starts increasing. The point P were the total cost in the minimum is called the Economic life of the asset. To solve problems under replacement, we consider the basics of interest formula.

Present worth factor denoted by (P/F, i,n). If an amount P is invested now with amount earning interest at the rate i per year, then the future sum (F) accumulated after n years can be obtained.

P - Principal sum at year Zero
F - Future sum of P at the end of the n^{th} year
i - Annual interest rate
n - Number of interest periods.

Then the formula for future sum $F = P (1 + i)^n$

$P = F/(1 +i)^n$ = Fx (present worth factor)

If A is the annual equivalent amount which occurs at the end of every year from year one through n years is given by

$$A = \frac{P \times i(1+i)^n}{(1+i)^n - 1}$$

$$= P (A / P, i, n)$$

= P x equal payment series capital recovery factor

Example:

A firm is considering replacement of an equipment whose first cost is Rs. 1750 and the scrap value is negligible at any year. Based on experience, it is found that maintenance cost is zero during the first year and it increases by Rs. 100 every year thereafter.

(i) When should be the equipment replaced if

a) i = 0%
b) i = 12%

Solution :

Given the first cost = Rs 1750 and the maintenance cost is Rs. Zero during the first years and then increases by Rs. 100 every year thereafter. Then the following table shows the calculation.

Calculations to determine Economic life

(a) First cost Rs. 1750 Interest rate = 0%

End of year (n)	Maintenance cost at end of year	Summation of maintenance	Average cost of maintenance through the	Average first cost if replaced at the given	Average total cost through the given year

		Cost	given year	year and	
A	**B (Rs)**	**C (Rs)**	**D (in Rs)**	**E (Rs)**	**F (Rs)**
		C = ΣB	**C/A**	$\frac{\mathbf{1750}}{\mathbf{A}}$	**D + E**
1	0	0	0	1750	1750
2	100	100	50	875	925
3	200	300	100	583	683
4	300	600	150	438	588
5	400	1000	200	350	550
6	**500**	**1500**	**250**	**292**	**542**
7	600	2100	300	250	550
8	700	2800	350	219	569

The value corresponding to any end-of-year (n) in Column F represents the average total cost of using the equipment till the end of that particulars year.

In this problem, the average total cost decreases till the end of the year 6 and then it increases. Hence the optimal replacement period is 6 years ie the economic life of the equipment is 6 years.

(e) When interest rate i = 12%

When the interest rate is more than 0% the steps to get the economic life are summarized in the following table.

Calculation to determine Economic life
First Cost = Rs. 1750 Interest rate = 12%

End of year (n)	**Maintenance cost at end of years**	**(P/F,12v,n)**	**Present worth as beginning of years 1 of maintenance costs**	**Summation of present worth of maintenance costs through the given year**	**Present simulator maintenance cost and first cost**	**(A/P, 12%,n) $=\frac{i(1+i)^n}{(1+i)^n-1}$ G**	**Annual equipment total cost through the giver year**
A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
	B (iR)	**C =** $\frac{\mathbf{1}}{\mathbf{(1+12/100)^n}}$	**BxC**	**ΣD**	**E+ Rs. 1750**		**FxG**
1	0	0.8929	0	0	1750	1.1200	1960
2	100	0.7972	79.72	79.72	1829.72	0.5917	1082.6
3	200	0.7118	142.36	222.08	1972.08	0.4163	820.9
4	300	0.6355	190.65	412.73	2162.73	0.3292	711.9
5	400	0.5674	226.96	639.69	2389.69	0.2774	662.9
6	500	0.5066	253.30	892.99	2642.99	0.2432	642.7
7	600	0.4524	271.44	1164.43	2914.430	0.2191	638.5
8	700	0.4039	282.73	1447.16	3197.16	0.2013	680.7

Identify the end of year for which the annual equivalent total cost is minimum in column. In this problem the annual equivalent total cost is minimum at the end of year hence the economics life of the equipment is 7 years.

5.3.5. Simple probabilistic model for items which completely fail

Electronic items like bulbs, resistors, tube lights etc. generally fail all of a sudden, instead of gradual failure. The sudden failure of the item results in complete breakdown of the system. The system may contain a collection of such items or just an item like a single tube-light. Hence we use some replacement policy for such items which would minimize the possibility of complete breakdown. The following are the replacement policies which are applicable in these cases.

i) Individual replacement policy :

Under this policy, each item is replaced immediately after failure.

ii) Group replacement policy :

Under group replacement policy, a decision is made with regard the replacement at what equal internals, all the item are to be replaced simultaneously with a provision to replace the items individually which fail during the fixed group replacement period.

Among the two types of replacement polices, we have to decide which replacement policy we have to follow. Whether individual replacement policy is better than group replacement policy. With regard to economic point of view. To decide this, each of the replacement policy is calculated and the most economic one is selected for implementation.

Exercise :

1. List and explain different types of maintenance
2. Discuss the reasons for maintenance.
3. Distinguish between breakdown maintenance and preventive maintenance.
4. Distinguish between individual and group replacement polices.
5. A firm is considering replacement of an equipment whose first cost is Rs.4000 and the scrap value is negligible at the end of any year. Based on experience, it has been found that the maintenance cost is zero during the first year and it is Rs.1000 for the second year. It increase by Rs.300 every years thereafter.
   a) When should the equipment be replace if i = 0%
   b) When should the equipment be replace if i = 12%
   Ans . a) 5 years b) 5 years

6. A company is planning to replace an equipment whose first cost is Rs.1,00,000. The operating and maintenance cost of the equipment during its first year of operation is Rs.10,000 and it increases by Rs. 2,000 every year thereafter. The release value of the equipment at the end of the

first year of its operation is Rs.65,000 and it decreases by Rs.10,000 every year thereafter. Find the economic life of the equipment by assuming the interest rate as 12%.

[Ans : Economic life = 13 years and the corresponding annual equivalent cost = Rs. 34,510]

7. The following table gives the operation cost, maintenance cost and salvage value at the end of every year of machine whose purchase value is Rs. 12,000. Find the economic life of the machine assuming.
   a) The interest rate as 0%
   b) The interest rate as 15%

End of year	**Operation cost at the end of year (Rs)**	**Maintenance cost at the end of year (Rs)**	**Salvage value at the end of year (Rs)**
1	2000	2500	8000
2	3000	3000	7000
3	4000	3500	6000
4	5000	4000	5000
5	6000	4500	4000
6	7000	5000	3000
7	8000	5500	2000
8	9000	6000	1000

Ans :

a) Economic life of the machine = 2 years
b) Economic life of the machine = 2 years

INVENTORY

Inventory is a stock of physical assets, having some economic value, which can be either in the form of material, men or money. Inventory is also called as an idle resource as long as it is not utilized. Inventory may be regarded as those goods which are procured, stored and used for day to day functioning of the organization.

Inventory can be in the form of physical resource such as raw materials, semi-finished goods used in the process of production, finished goods which are ready for delivery to the consumers, human resources, or financial resources such as working capital etc.

Inventories are viewed as a large potential risk rather than as a measure of wealth. The concept of inventories at present has necessitated the use of scientific techniques in the inventory management called as inventory control.

Thus, inventory control is the technique of maintaining stock items at desired levels. In other words, inventory control is the means by which material of the correct quality and quantity is made available as and when it is needed with due regard to economy in the holding cost, ordering costs, setup costs, production costs, purchase costs and working capital.

Inventory Management answers two questions viz. How much to order? and when to order?

OBJECTIVES OF MAINTINING INVENTORY

- To meet unexpected demand
- To achieve return on investment
- To order largest quantities of goods, components or materials from the suppliers at advantageous prices
- To provide reasonable customer service through supplying most of the requirements from stock without delay
- To avoid economically impractical and physically impossible delivering/getting right amount of stock at right time of required
- To maintain more work force levels
- To facilitate economic production runs
- To advantage of shipping economies
- To smooth seasonal or critical demand
- To facilitate the intermittent production of several products on the same facility
- To make effective utilization of space and capital
- To meet variations in customer demand
- To take the advantage of price discount and to hedge against price increases

FACTORS INVOLVED IN INVENTORY PROBLEM ANALYSIS

- Relevant inventory costs
- Demand for inventory items
- Replenishment lead time
- Length of planning period
- Constraint on the inventory system

INVENTORY COST COMPONENTS

1. **Purchase Cost:** Actual cost to be paid for procurement or manufacturing the items. May be independent or dependent on order quantity.
2. **Carrying (or holding) Cost:** It includes cost incurred on (i) storage cost for rent paid for the warehouse space, (ii) inventory handling cost for payment of salaries, (iii) insurance cost against fire or other form of damage, (iv) opportunity cost of the money invested in inventory, (v) obsolescence costs, deterioration costs, lost costs, (vi) depreciation, etc. It can be calculated on the basis of carrying cost per item for a unit time or the change of value of rupee.
3. **Ordering Cost:** This includes (i) requisition cost of handling of invoices, stationary, payments, etc. (ii) cost of services which includes cost of mailing, telephone calls, transportation, and other follow up actions (iii) materials handling cost incurred in receiving, sorting and inspecting (iv) accounting and auditing, etc.

   When an item is produced in-house, ordering cost is referred as set-up cost, which includes both paperwork costs and the physical preparation costs.
4. **Shortage and customer-service cost:** The shortage can be viewed in two different ways:

   (i) Customers are ready to wait for the supply of items: no loss of sale but extra paperwork cost will not be exactly known.

   (ii) Customers are not ready to wait for supply of items: loss of customer goodwill leading to loss of sale. Loss of goodwill will be proportional to the length of the delay.

REPLENISHMENT LEAD TIME

Order cycle: this is the time period between two successive replenishments. It may be determined in one of the following two ways:

- *Continuous Review*: the number of units of an item on hand is known and an order of fixed size is placed every time the inventory level reaches at a pre-specified level, called order point or reorder level.
- *Periodic Review*: the orders are placed at equal intervals of time but the size of the order may vary depending on the inventory on hand as well as the demand at the time of review.

Lead time (Delivery Lag)

REPLENISHMENT ORDER SIZE DECISIONS

Large order size will have following advantages:

(i) Reduction in frequency of orders

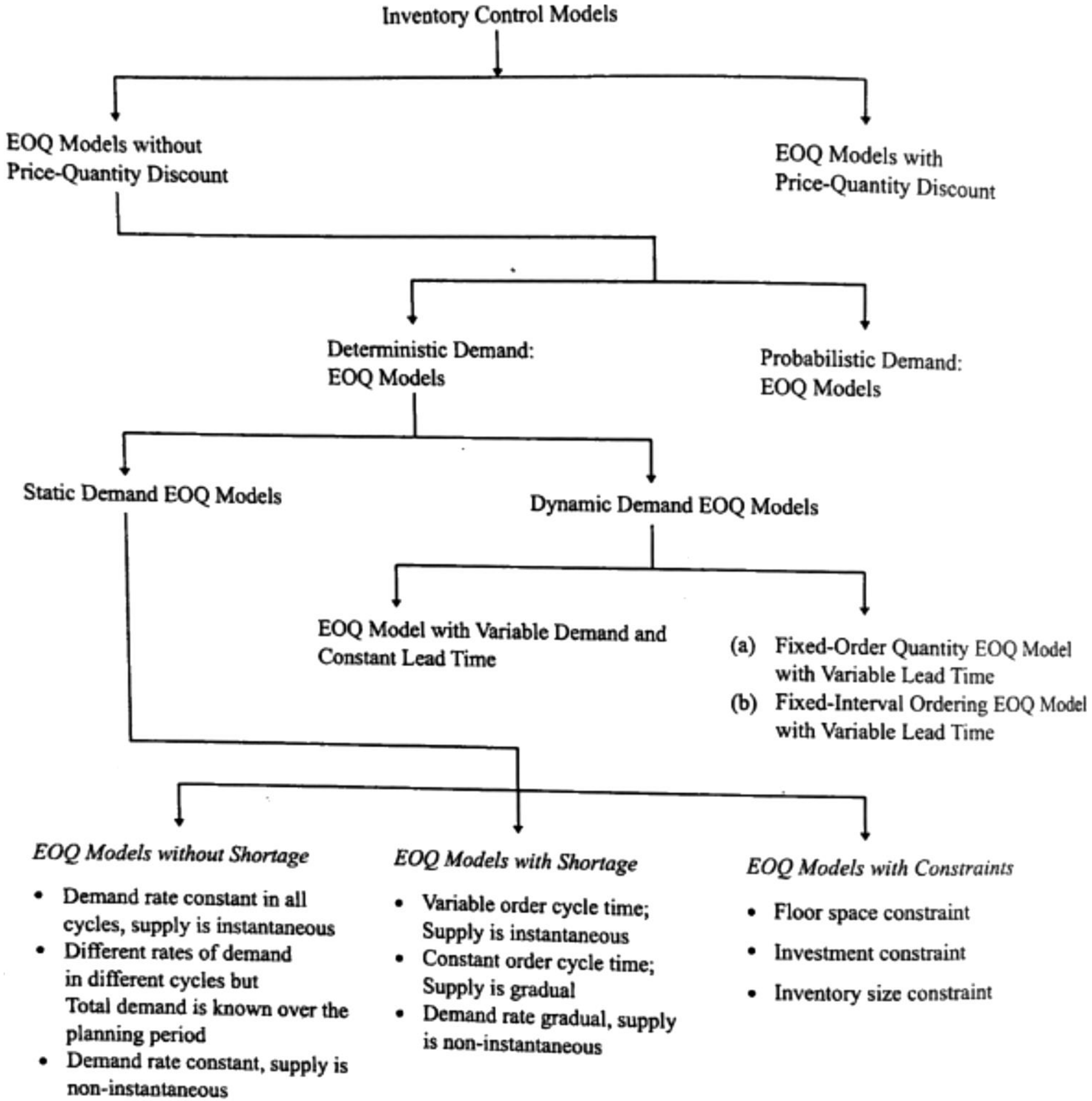
(ii) Reduction in total order cost

But large order size also lead to following disadvantage:

Increase in cycle stock inventory leading to increase in carrying cost (for excess inventory).

Hence, any decision on replenishment order size must be a trade-off between the ordering and carrying costs and the shortage cost. Such replenishment order is referred to as Economic Order Quantity (EOQ).

CLASSIFICATION OF DETERMINISTIC INVENTORY CONTROL MODELS



C = purchase cost or manufacturing cost per unit

C_0 = ordering cost per order

r = cost of carrying one rupee worth inventory expressed in the terms of percent of rupee value of inventory

C_h = $C \times r$ = cost of carrying one unit of inventory for a given length of time

C_s = Shortage cost per unit of inventory per unit time

D = Annual reqmt. (or demand) of an item

Q = order qty. per order

ROL = Re order level

LT = Lead Time

n = no. of ~~b~~order / time period

t = reorder cycle time

t_p = production period

r_p = production rate

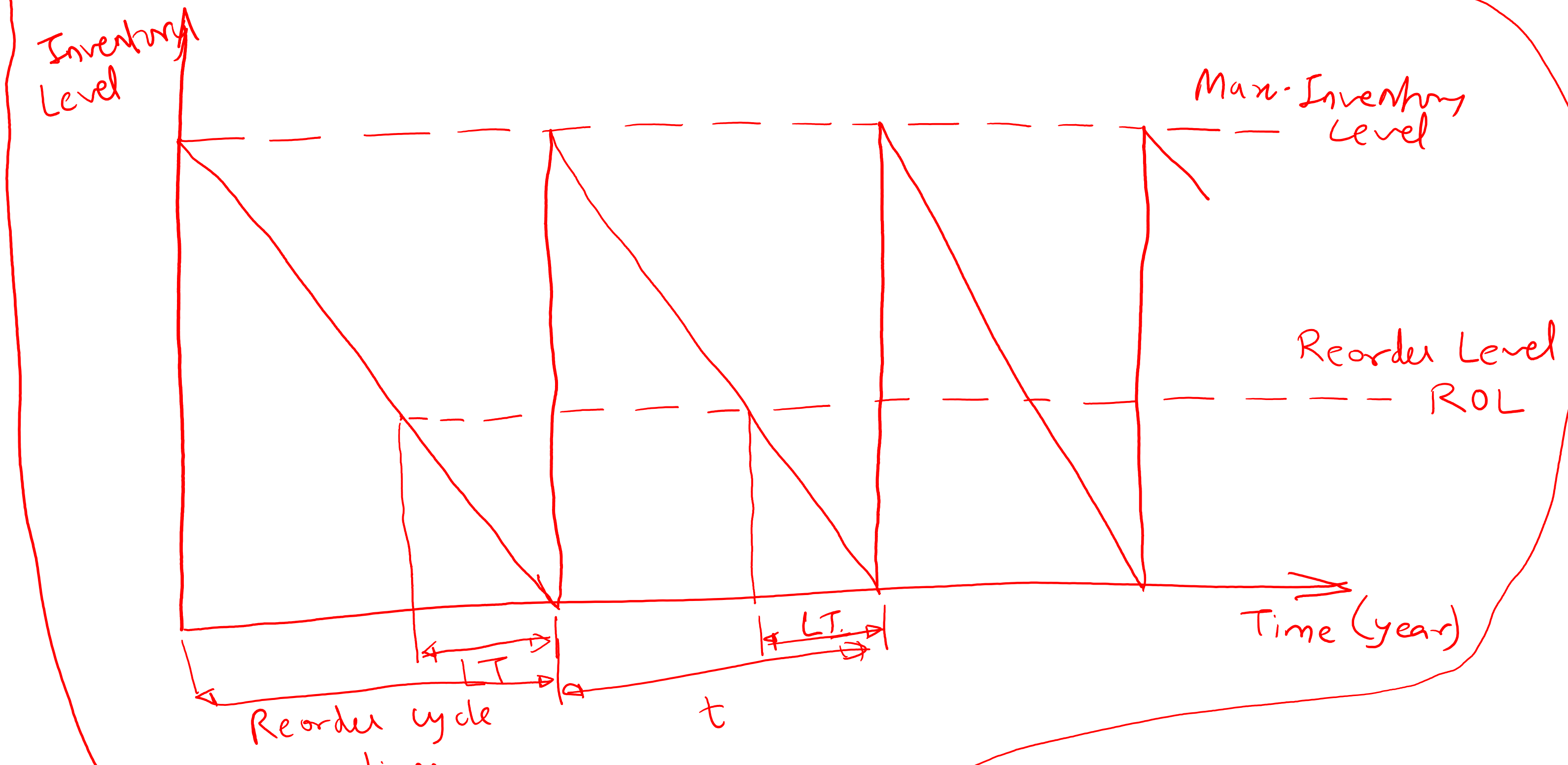
TC = Total inventory cost

TVC = Total variable inventory cost.

I. Single Item Inventory Control models without shortages.

(i) EOQ model with constant rate of Demand.

* Inventory involves only one type of items.
* The demand is known and it is constant.
* The inventory is replenished in single delivery for each order.



$t = \frac{Q}{D}$

Order qty replenished in one inventory cycle = Consumption of stock in one inventory cycle.

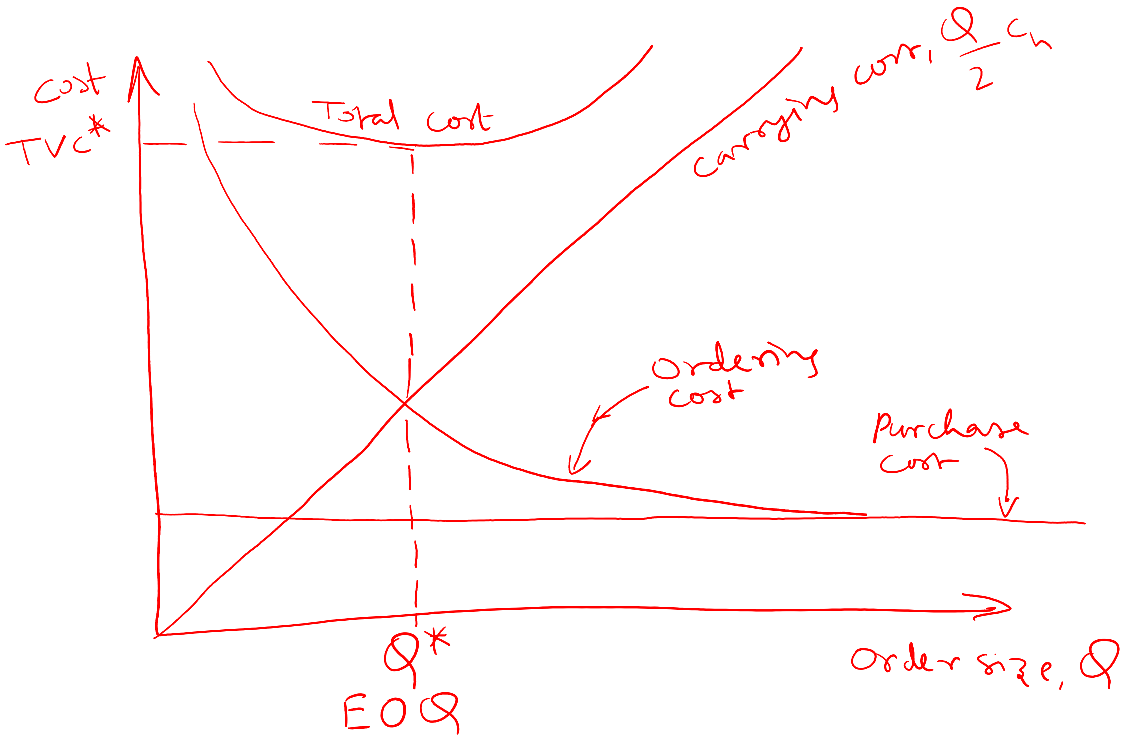
$Q = D \cdot t$

Total variable inventory cost incurred when an order of size Q is placed at the end of reorder cycle

$TVC = \{\text{Avg. Inventory level} \times (\text{carrying cost/unit/year})\} + \{\text{No. of orders placed per year} \times (\text{Ordering cost/order})\}$

$= \left(\frac{I_{max} + I_{min}}{2}\right) \times C_h + \frac{D}{Q} C_o$

$= \frac{Q}{2} \times C_h + \frac{D}{Q} C_o$



At EOQ :-

$$\frac{D}{Q} C_o = \frac{Q}{2} C_h$$

$$Q^2 = \frac{2 D C_o}{C_h}$$

$$\boxed{Q^* = \sqrt{\frac{2 D C_o}{C_h}}}$$

$$= \sqrt{\frac{2 \times \text{Annual Demand} \times \text{ordering cost}}{\text{Carrying cost}}}$$

Wilson or Harris lot size formula.

(i) optimal Interval of time t^* between two successive orders

$$t^* = \frac{Q^*}{D}$$

(ii) optimal No. of orders in a given period of time, N^*

$$N^* = \frac{\text{Annual Demand}}{\text{optimal order size}} = \frac{D}{Q^*} = D \times \frac{1}{\sqrt{\frac{2 D C_o}{C_h}}} = \sqrt{\frac{D C_h}{2 C_o}}$$

Optimal Total variable Inventory cost, TVC^*

Min

$$TVC^* = \frac{D}{Q^*} C_o + \frac{Q^*}{2} C_h$$

$$= D \cdot C_o \times \frac{1}{\sqrt{2DC_o/C_h}} + \sqrt{\frac{2DC_o}{C_h}} \cdot \frac{C_h}{2}$$

$$= \sqrt{2DC_oC_h}$$

Optimal Total Inventory cost

$$TC^* = D \cdot C + TVC^*$$

Q. 1 The production department of a company requires 3600 kg of raw material for manufacturing a particular item per year. It has been estimated that the cost of placing an order is Rs. 36 and the cost of carrying inventory is 25 percent of the investment in the inventories. The price is Rs. 10 per kg. Help the purchase manager to determine an ordering policy for raw material.

$D = 3600$ kg/yr $\quad C_o = $ Rs. 36/order $\quad C_h = 25\%$ of Rs. 10/kg = Rs. 2.50/kg/year

Optimal order size (EOQ) $= Q^* = \sqrt{\dfrac{2DC_o}{C_h}} = \sqrt{\dfrac{2 \times 3600 \times 36}{2.50}} = 321.99$ kg/order

optimal order cycle time $= t^* = \dfrac{Q^*}{D} = 0.894$ year

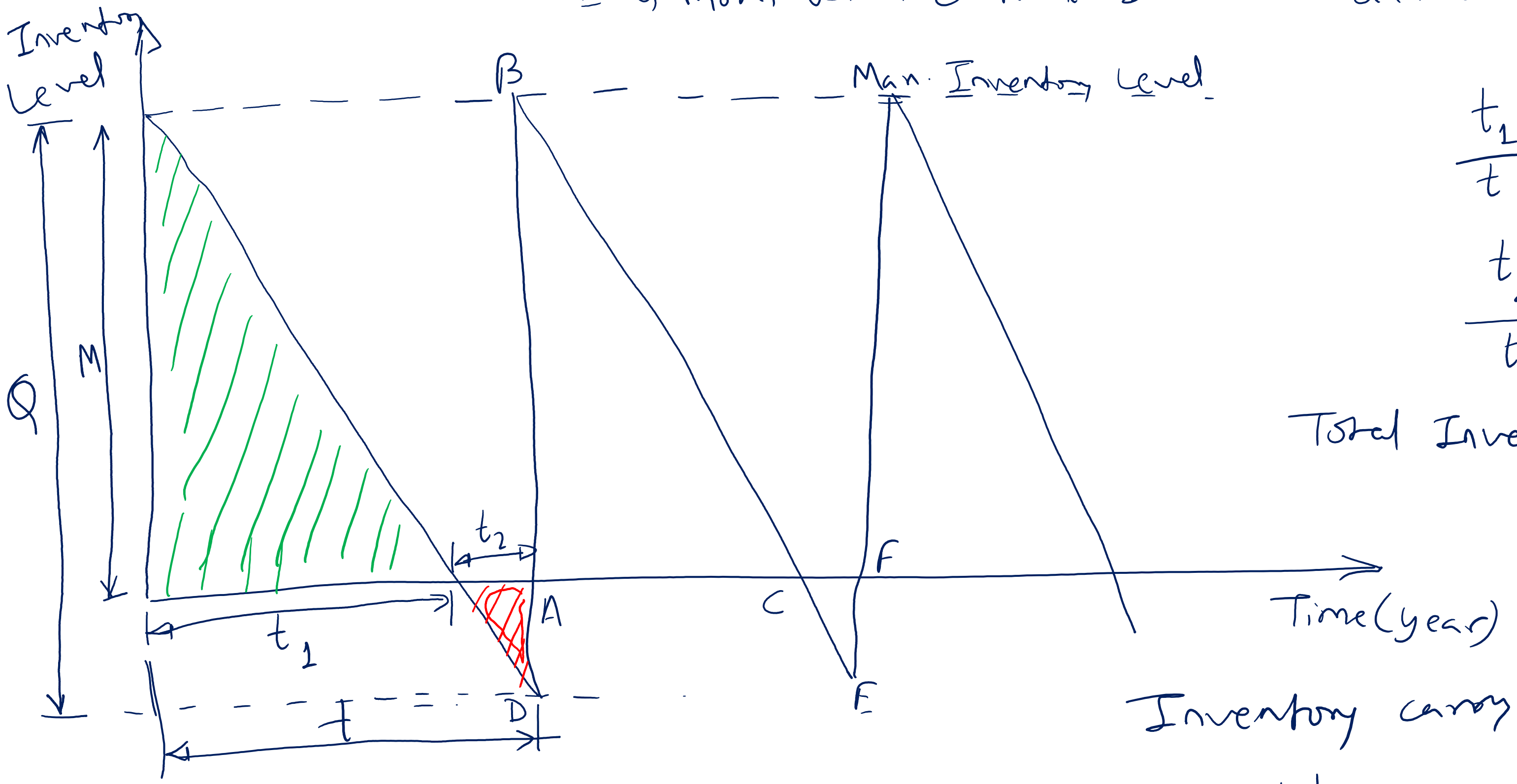
per year min. total variable inventory cost

$TVC^* = \sqrt{2DC_oC_h} = $ Rs. 804.98/yr.

per year min total inventory cost

$TC^* = TVC^* + D \cdot C = $ Rs. 804.98 + 3600 × 10 = Rs. 36,804.98/yr.

EOQ model with constant Demand Rate and with shortage



$$\frac{t_1}{t} = \frac{M}{Q} \quad \therefore \; \boxed{t_1 = \frac{M}{Q} \cdot t}$$

$$\frac{t_2}{t} = \frac{Q-M}{Q} \quad \therefore \; \boxed{t_2 = \frac{Q-M}{Q} \cdot t}$$

Total Inventory during time 't'

$= \text{area of } \Delta ABC = \frac{1}{2} M t_1$

Inventory carrying cost during time 't'

$= \frac{1}{2} c_h M t_1$

$t = t_1 + t_2$

Total shortage during time 't' $= \text{Area of } \Delta CFE$

$= \frac{1}{2}(Q-M)\, t_2$

Total Shortage cost during time 't' $= \frac{1}{2}(Q-M)\, t_2 \cdot C_s$

Ordering cost during time 't' $= C_o$

$\therefore$ Total inventory variable cost during time 't' $= \frac{1}{2} M t_1 C_h + \frac{1}{2}(Q-M) c_s t_2 + c_o$

Total avg. cost per unit time $= \frac{1}{t}\left[\frac{1}{2} c_h M t_1 + \frac{1}{2} c_s (Q-M) t_2\right] + \frac{C_o}{t} = \frac{1}{t}\left[\frac{1}{2} c_h M \cdot \frac{M}{Q} t\right.$

$$TVC = \frac{1}{t}\left[\frac{1}{2}C_h M \cdot \frac{M}{Q} t + \frac{1}{2}C_s (Q-M) \cdot \frac{Q-M}{Q} \cdot t\right] + \frac{C_o}{t}$$

$$= \frac{1}{t}\left[\frac{1}{2}C_h \frac{M^2}{Q} t + \frac{1}{2}C_s \frac{(Q-M)^2}{Q} t\right] + \frac{C_o}{t}$$

$$TVC = \frac{1}{2}C_h \frac{M^2}{Q} + \frac{1}{2}C_s \frac{(Q-M)^2}{Q} + C_o \cdot \frac{D}{Q}$$

TVC is a function of Q & M.

$\frac{\partial(TVC)}{\partial Q} = 0$ $\quad \frac{\partial^2(TVC)}{\partial Q^2} \Rightarrow$ positive qty $\quad$ TVC $\quad Q \Rightarrow$ Min TVC

$\frac{\partial(TVC)}{\partial M} = 0$ $\quad \frac{\partial^2(TVC)}{\partial M^2} \Rightarrow$ positive qty $\quad \Rightarrow \quad M \Rightarrow$ min TVC

$$Q^* = \sqrt{\frac{2DC_o}{C_h}\left\{\frac{C_h + C_s}{C_s}\right\}}$$ Economic Order Qty

$$M^* = \sqrt{\frac{2DC_o}{C_h}\left\{\frac{C_s}{C_h + C_s}\right\}}$$ optimal stock Level

$$TVC^* = \sqrt{2DC_oC_h\left\{\frac{C_s}{C_h + C_s}\right\}}$$ optimal Inventory variable cost.